

**INVESTIGATIONS IN
HIGHER DERIVATIVE FIELD THEORIES**

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By

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*TO
MY PARENTS*

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Chapter 1

Introduction

We know, in usual theories the Lagrangian is a function of fields and their first derivatives, known as first order systems. These systems were conceptualised from the time of Newton and successfully utilised for explaining various physical phenomena. But, theories with higher time derivatives of the fields became essential in some cases to make those theories renormalisable and free from divergences [1–3]. We call those theories as higher derivative theories. The higher derivative terms mainly occur as correction terms to the first order theories which play an important role during quantisation. Lagrangian theories with higher order derivatives are interesting in their own right. Since then, there appear numerous fields where the concepts of higher derivative theories have been used. Broadly speaking, in field theory [4–7], string theory [8], gravity [2, 9–11] and cosmology [12–14] higher derivative terms are frequently considered. Adding a higher derivative term can regularize the ultraviolet behaviour of the theory [1, 4]. These higher derivative terms can make the modified gravity renormalizable and even asymptotically free [2, 3]. People constructed $f(R)$ gravity where higher curvature terms were added to the Einstein-Hilbert action and opened a vast sector of research [2, 15–18]. For higher derivative gravity, the list is huge [2, 9, 10, 19–23]. higher derivative theories were also considered in the most exciting fields of recent theoretical physics like AdS/CFT correspondence which indicate the importance and relevance of considering higher derivative theories [14, 24, 25]. Due to these important applications of higher derivative theories, a systematic analysis of these theories have become necessary.

For physicists, studies of symmetries of a dynamical system is very important as many physical phenomena can be predicted just considering the symmetries. Gauge symmetry plays an important role in modern theoretical physics. Existence of gauge symmetries means not all degrees of freedoms are physically relevant. For this reason, abstraction of gauge symmetries to make a theory physically relevant has become very important. In physics, existence of a particular symmetry in the theory always refers to some conserved quantity. In fact, some

experimental results were predicted and benefited just considering the symmetries of the physical processes occurring during the experiment. Recent LHC results and their consequences are nothing but finding out the “broken symmetries”. For this reason, the importance of finding gauge symmetries should not be overlooked. In this thesis, we will focus on gauge symmetries and their consequences for higher derivative theories.

A systematic algorithm to extract the symmetries of higher derivative theories can serve as a useful contribution for the future analysis of these theories. For this, we need to follow some Hamiltonian formulation. In 1850, M. Ostrogradsky [26] gave the Hamiltonian formulation for higher derivative theories. But this formalism suffers from complications. The momenta, in this case, are defined via some non-trivial definition due to the higher derivative terms. All the higher derivatives of the fields are considered as independent variables and then Dirac’s Hamiltonian formulation is followed. But, it has been noticed, in some cases, that the appearance of the number of independent primary first class constraints do not match with the number of independent gauge symmetries ¹ [27]. For this reason it become necessary to have a well tested proper formulation to systematically abstract the gauge symmetries. This is the cornerstone of the present thesis.

To abstract the gauge symmetries, other than the Ostrogradsky approach [26, 28], we are going to follow the first order formalism [29–31] along with the powerful techniques developed by Banerjee et. al [32]. According to the first order formulation, the fields and their corresponding time derivatives are considered as independent variables to convert the higher derivative Lagrangian to a first order one. From this first order Lagrangian one can construct the Euler-Lagrange equations of motion and correspondingly develop a Hamiltonian formulation. But, transition to Hamiltonian formulation becomes complicated when there appears constraints. Constraints are actually some functions of the phase-space variables which prohibit to have the whole access of the phase-space. For this reason, constraint systems need careful analysis. From the Hamiltonian formulation one can further proceed for quantisation of the system. This quantisation procedure is known as canonical quantisation, formerly introduced by P. A. M. Dirac [33]. In this seminal work by Dirac, constrained systems and their systematic quantisation procedure are discussed. There is other literature available for further studies on constrained systems [34–39] particularly for the first order systems.

In the first order formulation, due to redefinition of the fields, extra constraints at the Lagrangian level appear which are incorporated via Lagrange multipliers. Now, these Lagrange multipliers are in fact treated as independent fields. Usual momenta definition is used which distinguishes this formulation from Ostrogradski method. Due to the standard momenta definitions, the calculations become easier and transparent. Constrained dynamics itself is a very interesting concept.

¹For usual theories it has been shown that the no. of independent gauge parameters equals the no. of independent primary first class constraints of the theory

All/some of the momenta in this case are not expressible with respect to velocities and hence the full phase space is not accessible. In this case the Lagrangian becomes singular and determinant of the Hessian matrix is equal to zero. Constraints are usually classified in the Dirac scheme as first class and second class. If their algebra closes, the set of constraints is first class, else it is second class. Consequently, the following possibilities may arise:

I. The original Lagrangian is singular but the additional constraints are all second class. The reduction of phase space may be done by implementing the second class constraints strongly provided we replace all the Poisson brackets(PB) by appropriate Dirac brackets(DB).

II. The original Lagrangian is singular and there are both second class and first class constraints among them. The second class constraints may be eliminated again by the DB technique. The first class constraints generate gauge transformations which are required to be further analysed. These constraints may yield further constraints and so on. The iterative process stops when no new constraints are generated.

The first class constraints generate gauge transformations. The gauge generator is defined as the linear combination of all the first class constraints. The coefficients of the constraints are called gauge parameters. Now, all the gauge parameters may or may not be independent of each other. To identify the independent gauge symmetries we apply the effective scheme developed by Banerjee and co-authors [32] which relates the gauge parameters and the Lagrange multipliers and suitability abstracts the independent gauge parameters. Once the independent gauge degrees of freedom are identified the gauge generator can easily be expressed with respect to these parameters. The number of independent gauge parameters refers to the number of independent degrees of freedom. This completes the process of finding out the gauge symmetries of a higher derivative system.

Gauge symmetric higher derivative theories appear in different contexts like the rigid relativistic particle models [30,31], the extended Maxwell-Chern-Simons theory [5], the quantum charged rigid membrane [40], the Regge-Teitelboim model in cosmology [41] etc and many more. Applications of this first order approach in different field of physics are carried out sequentially to provide the efficacy of the first order formalism

The first higher derivative model we are going to study is the massive relativistic particle model with curvature which is an extension of Polyakov's string model concept [42]. There he added, a scale invariant extrinsic curvature term to the usual relativistic particle model action whose influence in the infrared region determines the phase structure of the string theory. The particle model of this theory was introduced explicitly by Pisarski [6] where he considered the action in which the curvature of the path appeared along with the length coupled via a dimensionless constant. The model is important for polymer physics [43].

Also, Plyuschay has shown that the massless case can be interpreted as the description of either bosons or fermions depending on the value of the quantized parameter [44]. The Ostrogradskian Hamiltonian analysis of this model was subsequently carried out later by many authors in [27, 45–47]. M. S. Plyuschay is worthy to be mentioned [30, 31] for his contribution in relativistic particle models. The 2 + 1 dimensional version for the relativistic particle model with curvature and torsion was studied in [48]. Ramos et al showed that the massless relativistic particle model with curvature obeys W_3 algebra [45]. More literature on W_3 algebra can be found here [48–51].

In the first order formalism, the relativistic particle model with curvature is converted into a first order form by redefinition of the fields and their corresponding higher time derivatives. It is seen that the theory is constrained one. Preserving these primary constraints in time, the secondary constraints are found out for the model. After all the constraint combinatorics, we see that there are two independent primary first class constraints. Using these two first class constraints the gauge generator is constructed. It is seen that there is only one independent gauge parameter. Since there are two independent PFC we expected to have two independent gauge parameters. The mismatch also existed previously while considering the Ostrogradski approach but remained unnoticed for a long time [27]. The solution of this mismatch is clearly our finding. Also, the independent gauge symmetry is identified as reparametrisation invariance.

The massless version of this model that is constructed by keeping only the curvature term is shown to have two PFC along corresponding to two gauge degrees of freedom. One of them is the usual diffeomorphism while the other shows the W -symmetry. It is notable that this extra symmetry is revealed by casting the equations of motion in the Boussinesq form. By examining the most general transformations that preserve the structure of the Boussinesq Lax operators it was demonstrated that the symmetry group of the model satisfies W_3 algebra [45, 46]. W -algebras are nothing but extended Virasoro algebra in conformal field theories.

For quantisation of a theory with gauge invariance, BRST is a very powerful tool which also helps in the proof of the renormalizability and unitarity of gauge theories. This transformation, which is characterized by an infinitesimal, global and anticommuting parameter leaves the effective action as well as path integral of the effective theory invariant. The BRST quantisation for usual first order theories was first introduced in [52]. A series of works appeared thereafter in this direction [53]. The finite field dependent BRST (FFBRST) which considers the gauge parameters being finite and dependent on the fields have found many applications in various contexts. In gauge field theories the usual BRST symmetry has been generalized to make it finite and field dependent. The implementation of BRST symmetries for higher derivative theories is quite nontrivial and poses problems. In this context, therefore, a natural question arises regarding the application of BRST formalism to relativistic particle models. Indeed it is not

surprising that in spite of a considerable volume of research on relativistic particle models, this aspect remains unstudied.

We consider the relativistic particle model with curvature since it possesses gauge symmetry [54]. We construct the (anti-)BRST for the massive and massless models. The difficulties of applying BRST transformations to higher derivative theories are bypassed by working in the first order formalism developed in [29] instead of the conventional Ostrogradsky approach.

It is shown that the (anti-)BRST symmetry transformations for all the variables reproduce the diffeomorphism symmetry of the massive relativistic particle model including curvature. Furthermore, we also show that the massless particle model with rigidity yields both the diffeomorphism and W invariances. We explicitly demonstrate the W_3 algebra. For BRST transformations this algebra is shown for all variables, excluding the antighost. Exactly the same features are revealed, but now excluding the ghost variable instead of the antighost, for anti-BRST transformations. To get the complete picture, therefore, both BRST and anti-BRST transformations have to be considered. Further, we implement the concept of FFBRST transformation in the quantum mechanical relativistic particle model. The quantum mechanical version of FFBRST transformation [53] is called as finite coordinate-dependent BRST (FCBRST) transformation. We see that FCBRST transformation for the relativistic particle model is a symmetry of the action only, but not of the generating functional. Analogous to FFBRST the FCBRST transformation changes the Jacobian of path integral measure non-trivially. For an appropriate choice of finite coordinate dependent parameter FCBRST connects two different gauge-fixed actions within functional integration.

So far we have discussed the general method of abstracting gauge symmetries for the higher derivative theories for particle models only. A transition to field theories bring novel features even in the first derivative systems. It is natural to ask how our method in [29] works in the field theoretic models. Higher derivative theories also found their place in the field of electrodynamics long ago by Podolsky [7]. The usual Maxwell-Chern-Simon's(MCS) theory is a first order one which also violates parity conservation principle. The higher derivative extension of this MCS theory known as the extended Maxwell-Chern-Simon's(EMCS) theory was first introduced in [55]. Subsequently its Hamiltonian analysis in the perturbative approach was done by [5]. The higher derivative version of Chern-Simons model was also considered in [56]. We take the higher derivative model to analyse it in the first order approach [57]. Using the technique in [32] it is shown that the number of independent gauge transformation is one. This independent gauge transformation is identified as the $U(1)$ symmetry associated with the vector fields of the usual Maxwell theory.

The next higher derivative model we are going to consider is a membrane theory. In 1962, Dirac considered the electron to be a charged rigid membrane

in [58]. Based on this concept, a model was proposed in [59] by adding the extrinsic curvature of the world volume swept out by the membrane. Theories with extrinsic curvatures are frequently studied, but recent inclusion of these terms in some physically interesting models added an extra urgency to revisit the symmetry features of such theories. It is shown that because of the extrinsic curvature effects, there appears geometrical frustration when nematic liquid crystals are constrained to a curved surface [60]. Hamiltonian analysis of the membrane model of electron in the Ostrogradski formalism was done by Cordero et al. [61]. A series of work like domain wall formation [62], relativistic membranes [63], spiky membranes [64] and others [65] come into existence using this concept of membrane model. The geometry of these deformed relativistic membranes were also discussed in [66]. Even the extended version of this electron model was used in general relativity in which this particle is described as a source of the Kerr-Newman field [67] and Reissner-Nordstrom geometry [67]. Stability analysis of the model was studied in [67]. We take the model in [61] where the Lagrangian is expressed with respect to the local coordinates in the world volume. The model is higher derivative, although a surface term can be identified. We convert this higher derivative Lagrangian into a first order form and perform the Hamiltonian analysis [68]. It is shown that the model has only one gauge symmetry viz. the reparametrization invariance. The equation of motions for the higher derivative model matches with the first order model both at the Lagrangian and Hamiltonian levels.

After discussing the particle, field theoretic and membrane models we now take a model from gravity theory to show the efficacy of this first order formalism. We know, usually gravity is described in 4 dimensional pseudo Riemannian space-time. In 1977, Regge and Teitelboim introduced a model where gravity is considered to be induced on a hypersurface embedded in a higher dimensional Minkowski space [41]. Soon, the model was able to draw attention of the physics community as it is a manifestation of gravity from the string theoretic view. The model was reviewed at various stages by many authors [69–71]. Unlike gravity theories where metric is considered to be an independent variable, here the functions of the spacetime embeddings into a flat space are taken as independent variables. The canonical theory of gravity can be developed with respect to these induced variables. The configuration space in this case is infinite dimensional, called the superspace. In finite dimensions, the number of degrees of freedom gets restricted and the model correspondingly is described in the minisuperspace.

The Einstein Hilbert action with cosmological constant is considered to develop the minisuperspace version and apply the first order formalism [72, 73]. As in usual Einstein-Hilbert action a surface term can be isolated here also in this minisuperspace version. The action without the surface term was considered in [69–71] and Wheeler DeWitt equations [74] were constructed. Here, on the contrary, we keep the surface term since it has relation to entropy and develop

a Hamiltonian formulation. The model is a higher derivative one and we adapt the first order formalism. In this formalism we find out the gauge symmetry viz. the reparametrisation symmetry inherent in the system. The number of PFC is found out to be one. Final Dirac brackets are constructed by removing the pair of first class constraints by imposing appropriate gauge conditions. One of the gauge conditions is the usual cosmic gauge while the other one is newly proposed. Under these gauge conditions, the existing pair of first class constraints become second class which can be removed via the Dirac brackets. The final canonical pair is uplifted to the level of commutator. In quantum cosmology, Wheeler DeWitt equation is analogous to the Schrodinger equation [74]. So, to describe the quantum picture of the universe, the WDW equation is constructed and a subsequent match with existing literature is shown.

1.1 Outline of the thesis

The thesis starts with a review of Higher derivative theories and their canonical quantisation both in the Ostrogradski and the first order formalism. Thereafter, we will explicitly workout Hamiltonian formulation of various models e.g. the relativistic particle model with curvature [29, 54], the extended Maxwell-Chern-Simons model [57], the quantum charged rigid membrane [68], the Regge-Teitelboim model of cosmology [73] to show the effectiveness of this first order formalism. Also, while discussing these models we will come across various features of these models which automatically emerge out during the course of Hamiltonian formulation.

- *Chapter 2* discusses canonical quantisation of the constrained and unconstrained systems. This chapter briefly describes how to extract the gauge symmetries of a first class system. Also, transition to quantum picture is described formally. In this chapter, higher derivative systems are introduced and their canonical quantisation procedure in the Ostrogradski formalism is briefly described. Also, first order formalism is described.
- *Chapter 3* considers the relativistic particle model with curvature. Both its massive and massless versions are taken to abstract the symmetries. It is shown that there appears a mismatch in the number of independent primary first class constraints and number of independent gauge symmetries. The problem is solved by taking the first order approach and additionally considering a condition which appears only due to the higher derivative nature of the theory. The massive version shows the diffeomorphism while the massless version has W_3 symmetry along with the former one. The BRST quantisation for both the models is also discussed.

- *Chapter 4* takes the extended Maxwell-Chern-Simon theory which is a higher derivative field theoretic model. The gauge symmetry of this model is worked out. An exact mapping between the Hamiltonian gauge transformations and the $U(1)$ symmetries of the action has been established
- *Chapter 5* discusses the quantum charged rigid membrane model. The reparametrisation symmetry in the first order formalism is shown to be the symmetry of the action. The equations of motion for the higher derivative and first order system are shown to be equivalent to establish the validity of the first order approach.
- *Chapter 6* includes the Regge Teitelboim model of cosmology. Keeping the surface term, the model is analysed in the first order approach. The reparametrisation invariances are abstracted. By choosing appropriate gauge conditions the model is formally quantised in the reduced phase space. The Wheeler DeWitt equation is constructed and matching with the existing literature is shown for this model.
- *Chapter 7* contains the conclusions.

Chapter 2

Canonical quantisation

Information of any dynamical system can be described by real functions of time $q^i(t)$ and its time derivatives $\dot{q}^i(t)$. Using this collection which forms the configuration space, the state of any dynamical system can be specified. The starting of our discussion should be the action principle which led to the development of the Lagrangian and Hamiltonian formulation of classical mechanics. Using the action principle, we get the equations of motion commonly known as Euler-Lagrange equations of motion. Consider the action as function of $q(t)$ with fixed end points $q_1(t)$ and $q_2(t)$

$$S[q; t_1, t_2] = \int_{t_1}^{t_2} L(q^i, \dot{q}^i, t) dt. \quad (2.1)$$

According to action principle, *the trajectory is the extremum of this action*. The $L(q, \dot{q})$ is a real function of time, called the Lagrangian. The extremum condition says

$$\frac{\delta S}{\delta q^i(t)} = 0. \quad \delta q(t_1) = \delta q(t_2) = 0. \quad (2.2)$$

Consequently the equations of motion can be written as

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i(t)} - \frac{\partial L}{\partial q^i} = 0. \quad (2.3)$$

These second order differential equations are known as Euler-Lagrange(EL) equations. However, these differential equations can be of higher order for the Lagrangians containing higher time derivatives of q 's. This entire chapter is devoted for the first order systems.

One can convert these second order equations to first order by shifting from the configuration space (q^i, \dot{q}^i) to phase space (q^i, p_i) . These p_i are known as momenta and defined by

$$p_i = \frac{\partial L}{\partial \dot{q}^i} \quad (2.4)$$

Sometimes, all or some of the momenta are not expressible with respect to velocities. These equations lead to constraints. Constraints are some functions of the phase space and need special care during the process of quantisation. Below we describe quantisation of unconstrained and constrained systems separately.

2.1 Unconstrained systems

To see the evolution of the system, we define a function using the phase space variables called Hamiltonian via the Legendre transformation as

$$H(q^i, p_i) = p_i \dot{q}^i - L(q, \dot{q}). \quad (2.5)$$

Using the same boundary conditions as in (2.2) we get the Hamilton's equations of motion

$$\dot{p}_i = \frac{\partial H}{\partial q^i}; \quad \dot{q}^i = \frac{\partial H}{\partial p^i} \quad (2.6)$$

These Hamilton's equations can also be easily obtained using Poisson brackets. For two functions $A(q, p)$ and $B(q, p)$ of the phase space variables the Poisson bracket(PB) is a bilinear operation defined as (see appendix for properties of Poisson Brackets)

$$\{A(q_i, p^i), B(q_i, p^i)\} = \frac{\partial A(q_i, p^i)}{\partial q_j} \frac{\partial B(q_i, p^i)}{\partial p^j} - \frac{\partial A(q_i, p^i)}{\partial p^j} \frac{\partial B(q_i, p^i)}{\partial q_j}. \quad (2.7)$$

With these the PBs for the basic variables turn out to be

$$\{q^i, p_j\} = \delta_j^i. \quad (2.8)$$

The Hamilton's EOM now become as

$$\dot{q}^i = \{q^i, H\}; \quad \dot{p}_i = \{p_i, H\} \quad (2.9)$$

We can move to the quantum picture of the system where the Hilbert space is spanned by these canonical variables (q^i, p_i) lifted to the operators (\hat{q}^i, \hat{p}_i) and the PBs are converted to commutators via the following relation

$$\{.,.\} \rightarrow \frac{1}{i\hbar}[.,.] \quad (2.10)$$

All the dynamical variables in the phase space are the operators in the chosen Hilbert space. This method of quantisation is known as canonical quantisation and was introduced by P. A. M. Dirac [33].

2.2 Constrained systems

From (2.3) we can write

$$\ddot{q}^i \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} = \frac{\partial L}{\partial q^j} - \dot{q}^i \frac{\partial^2 L}{\partial q^i \partial \dot{q}^j} \quad (2.11)$$

From this equation we see that if the determinant of the matrix $\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$ is zero the accelerations \ddot{q}^i are not uniquely determined which signals that all the momenta in (2.4) are not expressible with respect to the velocities. This means the system is constrained one. For constrained systems the method of canonical quantisation is more complicated. We write the constraint equations in a general form as

$$\phi_m(q, p) \approx 0. \quad (m = 1, \dots, M) \quad (2.12)$$

We got these equations from the definitions of the momenta itself. These constraints are known as *primary constraints*. The sign \approx refers to the *weak condition* which means that they should be put to zero only after evaluation of the PBs. We can add constraints to the Hamiltonian (2.5) and correspondingly see the evolution. Therefore, the definition (2.5) is not unique for the constraint systems. For a constraint system it is better to use the total Hamiltonian defined by

$$H_T = H_{can} + \lambda^m \phi_m. \quad (2.13)$$

Here H_{can} is known as the canonical Hamiltonian defined by (2.5) and λ^m are the Lagrange multipliers enforcing the primary constraints.

Once we get the primary constraints we can see the time variation of the constraints as

$$\psi_m = \{\phi_m, H_T\}. \quad (2.14)$$

These ψ_m can be zero identically or be linear combination of the other constraints. Another possibility can arise where they are some other functions of the phase space variables. Demanding this to be zero we get the secondary constraints. In this way preserving the secondary constraints in time we can get the tertiary constraints. At some point, the chain terminates and we have no more generation of new constraints. Once we have the whole list of the constraints, another important classification can be done based on their PB structure. In this case they are defined as either first class or second class.

- *First class* constraints are those which have a closed Poisson algebra. Effectively this implies that they have weakly vanishing PB among themselves. First class constraints are the generators of the gauge transformation.

– *Second class* constraints are those which do not fall in the above class.

This classification is very important for Hamiltonian formulation. It is to be mentioned that the number of independent primary first class constraints are equal to the number on independent gauge symmetries. The existence of the first class constraints in the system means there are some redundant degrees of freedom. To quantise the theory, in the reduced phase space, we have to remove all the constraints. First we remove the second class constraints by defining the Dirac brackets(DB) analogous to Poisson brackets as

$$\{A, B\}_D = \{A, B\} - \{A, \chi_i\} \Delta_{ij}^{-1} \{\chi_j, B\}. \quad (2.15)$$

Here χ_i are the set of all second class constraints and $\Delta_{ij} = \{\chi_i, \chi_j\}$. All the second class constraints can be put strongly to zero and in the quantum case they are treated as operator identities. Also, hereafter, whenever the PB appears they should be replaced by DB defined in (2.15). With the first class constraints remaining in the system we can proceed to abstract the gauge symmetries by defining the gauge generator as the linear combination of all the first class constraints (φ_a) written as

$$\mathcal{G} = \epsilon^a \varphi_a. \quad (2.16)$$

Here ϵ^a are known as the gauge parameters. All the gauge parameters in the system may or may not be independent. The independent gauge parameters actually signifies the gauge symmetries in the system. Following the algorithm of [32] we can express the dependent gauge parameters in terms of the independent set using the conditions

$$\delta \lambda^{a_1} = \frac{d\epsilon^{a_1}}{dt} - \epsilon^a (V_a^{a_1} + \lambda^{b_1} C_{b_1 a}^{a_1}) \quad (2.17)$$

$$0 = \frac{d\epsilon^{a_2}}{dt} - \epsilon^a (V_a^{a_2} + \lambda^{b_1} C_{b_1 a}^{a_2}) \quad (2.18)$$

The indices $a_1, b_1 \dots$ refer to the primary first class constraints while the indices $a_2, b_2 \dots$ correspond to the secondary first class constraints. Indices without any suffix like $a, b \dots$ comprise both the primary and secondary set. The coefficients $V_a^{a_1}$ and $C_{b_1 a}^{a_1}$ are the structure functions of the involutive algebra, defined as

$$\{H_c, \varphi_a\} = V_a^b \varphi_b \quad (2.19)$$

$$\{\varphi_a, \varphi_b\} = C_{ab}^c \varphi_c \quad (2.20)$$

and λ^{a_1} are the Lagrange multipliers(associated with the primary first class constraints) appearing in the expression of the total Hamiltonian (2.13).

Solving (2.17) it is possible to choose a_1 independent gauge parameters from the set ϵ^a and express \mathcal{G} of (2.16) entirely in terms of them. Once we get the independent gauge parameters (ϵ^a), we can find out the gauge transformation of the fields as

$$\delta q^i = \epsilon^a \{q^i, \varphi_a\} \quad (2.21)$$

Now we have to get rid of these first class constraints φ_a . This can be done by introducing gauge conditions. Gauge conditions are some ad hoc relations which make the first class constraints second class. Once the first class constraints become second class they can be removed by Dirac brackets. In the reduced phase space we can identify the canonical pairs and proceed for quantisation by promoting these BDs as commutators as in (2.10). This completes the canonical quantisation for constraint systems.

2.3 Higher derivative theories

By higher derivative theories we specifically refer to those theories where higher time derivative of the fields appear. We begin with a general higher derivative theory given by the Lagrangian

$$L = L(x, \dot{x}, \ddot{x}, \dots, x^{(\nu)}) \quad (2.22)$$

where $x = x_n (n = 1, 2, \dots, \nu)$ are the coordinates and $\dot{}$ means derivative with respect to time. ν -th order derivative of time is denoted by $x^{(\nu)}$. Below we discuss the canonical formulation of the higher derivative theories in the Ostrogradski and first order formalisms.

2.3.1 Ostrogradski formalism

In the Ostrogradski method [26], for the higher derivative Lagrangian (2.22), the Euler Lagrange EOM is written as

$$\sum_{l=0}^{\nu} (-1)^l \frac{d^l}{dt^l} \left(\frac{\partial L(x, \dot{x}, \ddot{x}, \dots, x^{(\nu)})}{\partial x_{n,l}} \right) = 0 \quad (2.23)$$

In this approach, all the fields along with their corresponding higher derivatives are considered as independent fields. Correspondingly, two types of momenta are defined. To begin the Hamiltonian formulation, the momenta

are defined as,

$$\begin{aligned} P_i &= \frac{\partial L}{\partial q_i^{(m)}}, \\ p_{i,m-1} &= \frac{\partial L}{\partial q_i^{(m)}} - \frac{d}{dt} \left(\frac{\partial L}{\partial q_i^{(m+1)}} \right); \quad m = 1, 2, \dots, n-1 \end{aligned} \quad (2.24)$$

Here the canonical pairs are $(q_i, P_{i,m-1})$ and $(q_i^m, p_{i,m})$ and thus we get the Hamiltonian as

$$H = \sum_{m=1}^{n-1} p_{i,m} q_i^{(m)} + P_i \dot{q}_i^{(n-1)} - L. \quad (2.25)$$

Once the Hamiltonian is defined the Hamiltonian equations of motion can be found in the usual way as (2.9).

2.3.2 The first order formalism

The Hamiltonian formulation of the theory may be conveniently done by a variant of Ostrogradskii method commonly known as the first order formalism. The crux of the method consists in embedding the original higher derivative theory to an effective first order theory. We define the variables $q_{n,\alpha}$ ($\alpha = 1, 2, \dots, \nu - 1$) as

$$\begin{aligned} q_{n,1} &= x_n \\ q_{n,\alpha} &= \dot{q}_{n,\alpha-1}, (\alpha > 1) \end{aligned} \quad (2.26)$$

This leads to the following Lagrangian constraints

$$q_{n,\alpha} - \dot{q}_{n,\alpha-1} = 0, (\alpha > 1) \quad (2.27)$$

which must be enforced by corresponding Lagrange multipliers. The auxiliary Lagrange function of this extended description of the system is given by

$$\begin{aligned} L^*(q_{n,\alpha}, \dot{q}_{n,\alpha}, \lambda_{n,\beta}) &= L(q_{n,1}, q_{n,2}, \dots, q_{n,\nu-1}, \dot{q}_{n,\nu-1}) + \\ &\sum_{\beta=2}^{\nu-1} (q_{n,\beta} - \dot{q}_{n,\beta-1}) \lambda_{n,\beta}, \end{aligned} \quad (2.28)$$

where $\lambda_{n,\beta}$ ($\beta = 2, \dots, \nu - 1$) are the Lagrange multipliers. If we consider these multipliers as independent fields then the Lagrangian L^* becomes first order to which the well known methods of Hamiltonian analysis for first order systems apply. The momenta canonically conjugate to the degrees of

freedom $q_{n,\alpha}$, ($\alpha = 1, 2, \dots, \nu - 1$) and $\lambda_{n,\beta}$ ($\beta = 2, \dots, \nu - 1$) are defined, respectively, by,

$$p_{n,\alpha} = \frac{\partial L^*}{\partial \dot{q}_{n,\alpha}}, \quad \pi_{n,\beta} = \frac{\partial L^*}{\partial \dot{\lambda}_{n,\beta}}. \quad (2.29)$$

These immediately lead at least to the following primary constraints,

$$\Phi_{n,\beta} \approx 0, \quad \pi_{n,\beta} \approx 0, \quad \beta = 2, \dots, \nu - 1, \quad (2.30)$$

where

$$\Phi_{n,\beta} \equiv p_{n,\beta-1} + \lambda_{n,\beta}, \quad \beta = 2, \dots, \nu - 1. \quad (2.31)$$

Note that depending on the situation whether the original Lagrangian L is singular there may be more primary constraints.

Let us first assume that the original Lagrangian L is regular. Then there are no more primary constraints. Now the basic non-trivial Poisson brackets are

$$\{q_{n,\alpha}, p_{m,\alpha'}\} = \delta_{nm}\delta_{\alpha\alpha'}, \quad \{\lambda_{n,\beta}, \pi_{m,\beta'}\} = \delta_{nm}\delta_{\beta\beta'}, \quad (2.32)$$

Consequently the primary constraints obey the algebra

$$\{\Phi_{n,\beta}, \Phi_{m,\beta'}\} = 0, \quad \{\Phi_{n,\beta}, \pi_{m,\beta'}\} = \delta_{nm}\delta_{\beta\beta'}, \quad \{\pi_{n,\beta}, \pi_{m,\beta'}\} = 0, \quad (2.33)$$

implying that $\pi_{n,\beta}$ and $\Phi_{n,\beta}$ are second class constraints. Now, the canonical Hamiltonian of the modified system(2.28) can be written according to the usual prescription as,

$$H_C = \sum_n \sum_{\alpha=1}^{\nu-1} \dot{q}_{n,\alpha} p_{n,\alpha} + \sum_n \sum_{\beta=2}^{\nu-1} \dot{\lambda}_{n,\beta} \pi_{n,\beta} - L^*, \quad (2.34)$$

We define the total Hamiltonian (H_T) by adding linear combinations of the primary constraints (2.30),

$$H_T = H_C + u_{n,\beta} \pi_{n,\beta} + v_{n,\beta} \Phi_{n,\beta} \quad (2.35)$$

where $u_{n,\beta}$ and $v_{n,\beta}$ are Lagrange multipliers. From (2.33) we found that the constraints are second class. Thus preserving the primary constraints in time we will be able to fix the multipliers $u_{n,\beta}$ and $v_{n,\beta}$. Fixing these and after some simplifications we find that

$$H_T(q_{n,\alpha}, p_{n,\alpha}; \lambda_{n,\beta}) = \bar{H}_0(q_{n,\alpha}, p_{n,\nu-1}) - \sum_n \sum_{\beta=2}^{\nu-1} \lambda_{n,\beta} q_{n,\beta}. \quad (2.36)$$

$$\bar{H}_0(q_{n,\alpha}, p_{n,\nu-1}) = \sum_n \dot{q}_{n,\nu-1} p_{n,\nu-1} - L(q_{n,\alpha}, \dot{q}_{n,\nu-1}). \quad (2.37)$$

The presence of the second class constraints (2.30) imply that the phase space degrees of freedom $(q_{n,\alpha}, p_{n,\alpha}; \lambda_{n,\beta}, \pi_{n,\beta})$ are not all independent. If we replace the Poisson brackets by Dirac brackets the constraints (2.30) can be strongly implemented. This enables us to eliminate the nondynamical sector $(\lambda_{n,\beta}, \pi_{n,\beta})$ from the phase space variables. It is to be noted that the no. of second class constraints equals the no. of nondynamical variables eliminated from the phase space. Straightforward calculations show that the DBs between the remaining phase space variables are the same as the corresponding PBs. The total Hamiltonian now becomes

$$H_T(q_{n,\alpha}, p_{n,\alpha}; \lambda_{n,\beta}) = \bar{H}_0(q_{n,\alpha}, p_{n,\nu-1}) + \sum_n \sum_{\beta=2}^{\nu-1} p_{n,\beta-1} q_{n,\beta} . \quad (2.38)$$

Since the original Lagrangian system is replaced by the first order theory (2.28) the algorithm of [32] can be readily applied. For the conventional first order theories this completes the picture. The situation for higher order theories is, however, different. This is because of the new constraints (2.27) appearing in the effective first order lagrangean (2.28). Owing to these we additionally require

$$\delta q_{n,\alpha} - \frac{d}{dt} \delta q_{n,\alpha-1} = 0, (\alpha > 1) \quad (2.39)$$

These conditions may reduce the number of independent gauge parameters further. Thus the number of independent gauge parameters is, in general, less than the number of independent primary first class constraints. Whereas, in usual theories the number of independent primary first class constraints is always equal to the number of independent primary first class constraints. This additional condition can play a remarkable role in determining the actual number of gauge symmetries present in the higher derivative systems making the difference from a genuine first order theory. As is evident here for first order theory the equation is not meaningful. While considering the algorithm for abstracting the gauge symmetry we must consider this relation to get the actual realisation of the theory.

The effective first order theory, obtained from the original higher order theory, therefore contains peculiarities. In this thesis we will illustrate this peculiarity using different examples.

2.4 Discussions

In this chapter we have shown how to canonically quantise a theory from the given action. If the theory is constrained one, then it needs special

attention. To handle these theories the well known Dirac's procedure has been explained. The gauge generator is constructed from the first class constraints. It has been shown how to extract the independent gauge symmetries using the technique developed in [32]. For the higher derivative theories, the situation however, is different where we depict the Ostrogradski formalism as well as the first order formalism. The first order formalism, which we are going to use throughout the thesis, is explained elaborately, even in the presence of constraints. The usual momenta definition is used to develop the Hamiltonian formulation for the first order approach.

Chapter 3

Relativistic particle model with curvature

In the last chapter we have discussed how to perform canonical formulations for the higher derivative theories. For that purpose we have developed a systematic algorithm to extract the independent gauge degrees of freedom. Our method is an extension of similar technique for the usual first order theories [32]. In the rest of the thesis we will illustrate our method for different particle and field theoretic models. In this chapter, we consider a particle model, known as the relativistic particle model with curvature, which is higher derivative in nature. The action of the relativistic particle model is defined by the arc length of the worldline [42]. They are important in physics in their own right and also behave as toy models to test various ideas which are useful for more general systems such as string and membranes. One can add the extrinsic curvature of the world line to this relativistic particle model which is generally known as the relativistic particle model with curvature [27,29–31]. The massive relativistic particle model with curvature [6] has the action ¹

$$S = -m \int \sqrt{\dot{x}^2} d\tau + \alpha \int \frac{((\dot{x}\ddot{x})^2 - \dot{x}^2 \ddot{x}^2)^{\frac{1}{2}}}{\dot{x}^2} d\tau \quad (3.1)$$

The Hamiltonian analysis in the Ostrogradski formalism shows that we have two independent primary first class constraints. Whereas, there is only one independent gauge degrees of freedom. Clearly, it is a mismatch with the existing literature (on usual non higher derivative theories) which says that the number of independent gauge symmetries are always equal to

¹contractions are abbreviated as $A^\mu B_\mu = AB$, $A^\mu A_\mu = A^2$. We consider the model in $3 + 1$ dimensions. So μ assumes the values $0, 1, 2, 3$. Also, the model is meaningful for $\alpha < 0$ [31]

the number of independent primary first class constraints. This paradox will be solved in the following sections. Also, the gauge generator with the appropriate number of independent gauge parameters will be constructed.

The action of the massless relativistic particle model can be obtained from (3.1) by putting $m = 0$. This model has been investigated by many authors [31, 45, 46]. It has five first class constraints, out of which two are primary. There is no second class constraint. So the gauge generator is constructed with five gauge parameters. The method of [32] shows that only two of them are independent. Also, there are two independent primary first class constraints. So, the case of mismatch in the number of independent PFCs and the independent gauge transformations does not occur here. Thus in the Ostrogradski approach, the no. of independent PFCs may or may not be equal to the no. of independent gauge degrees of freedom. In the following we will demonstrate this apparent mechanism of this arbitrariness. The massless relativistic particle model with curvature has a very interesting aspect. One of its two gauge symmetries may easily be identified as the usual diffeomorphism symmetry. The second symmetry was analysed only at the equations of motion level by introducing the Lax operators. It was demonstrated that his symmetry is nothing but W_3 symmetry [45]. In the following we will show that our method is capable of extracting both these symmetries from the action and W_3 algebra emerges systematically.

We will discuss the BRST symmetries of this massive and massless relativistic particle model with curvature. The BRST technique is a very powerful tool to quantise a theory with gauge invariance. The BRST symmetry transformations are characterized by an infinitesimal, global and anticommuting parameter that leaves the effective action as well as the path integral of the effective theory invariant. The BRST symmetries for higher derivative theories are still unexplored and quite nontrivial. Since the massive as well as the massless relativistic particle model with curvature poses gauge symmetries, their BRST study is necessary. The (anti-)BRST for the above models are constructed in [54]. For the massive relativistic particle model the (anti-)BRST transformations for all variables including ghost and anti-ghost exactly reproduce the diffeomorphism symmetry. The BRST transformation for the massless version also reproduce the W_3 -algebra but they exclude the anti-ghost variables. Whereas, the anti-BRST transformation in this case exclude the ghost fields. Other than this, the finite field dependent BRST is also considered for both these models. The quantum mechanical version which is known as the finite coordinate dependent BRST is a symmetry of the action only but not of the generating functional for this relativistic particle model.

3.1 Hamiltonian analysis and construction of the gauge generator

To convert the Lagrangian (3.1) into a first order one we introduce the new coordinates

$$q_1^\mu = x^\mu \quad q_2^\mu = \dot{x}^\mu \quad (3.2)$$

The Lagrangian in these coordinates has a first order form given by [30]

$$L = -m\sqrt{q_2^2} + \alpha \frac{((q_2\dot{q}_2)^2 - q_2^2\dot{q}_2^2)^{\frac{1}{2}}}{q_2^2} + q_0^\mu(\dot{q}_{1\mu} - q_{2\mu}) \quad (3.3)$$

where q_0^μ are the Lagrange multipliers that enforce the constraints

$$\dot{q}_{1\mu} - q_{2\mu} = 0 \quad (3.4)$$

Let $p_{0\mu}$, $p_{1\mu}$ and $p_{2\mu}$ be the canonical momenta conjugate to $q_{0\mu}$, $q_{1\mu}$ and $q_{2\mu}$ respectively. Then we immediately get the following primary constraints

$$\Phi_{0\mu} = p_{0\mu} \approx 0; \Phi_{1\mu} = p_{1\mu} - q_{0\mu} \approx 0 \quad (3.5)$$

and

$$\Phi_1 = p_2 q_2 \approx 0; \Phi_2 = p_2^2 q_2^2 + \alpha^2 \approx 0 \quad (3.6)$$

The first set of constraints (3.5) are an outcome of our extension of the original Lagrangian. The canonical Hamiltonian following from the usual definition is given by

$$H_C = m\sqrt{q_2^2} + q_{0\mu} q_2^\mu \quad (3.7)$$

The total Hamiltonian is

$$H_T = H_C + u_{0\mu} \Phi_{0\mu}^\mu + u_{1\mu} \Phi_{1\mu}^\mu + \xi^1 \Phi_1 + \xi^2 \Phi_2 \quad (3.8)$$

where $u_{0\mu}$, $u_{1\mu}$, ξ^1 and ξ^2 are as yet undetermined multipliers. Now, conserving the primary constraints we find that the multipliers $u_{0\mu}$ and $u_{1\mu}$ are fixed:

$$u_{0\mu} = 0 \quad \text{and} \quad u_{1\mu} = q_{2\mu} \quad (3.9)$$

Also new secondary constraints are obtained

$$\omega_1 = q_0 q_2 + m\sqrt{q_2^2} \approx 0; \quad \omega_2 = q_0 p_2 \approx 0 \quad (3.10)$$

The last constraint in (3.10) is obtained by assuming $q_2^2 \neq 0$ which follows from the structure of the Lagrangian (3.3). The total Hamiltonian now becomes

$$H_T = m\sqrt{q_2^2} + q_{0\mu} q_2^\mu + q_{2\mu} \Phi_{1\mu}^\mu + \xi^1 \Phi_1 + \xi^2 \Phi_2 \quad (3.11)$$

PBs among the constraints are given by

$$\begin{aligned}
\{\Phi_{0\mu}, \omega_1\} &= -q_{2\mu} \\
\{\Phi_{0\mu}, \omega_2\} &= -p_{2\mu} \\
\{\Phi_1, \omega_1\} &= -\omega_1 = 0 \\
\{\Phi_1, \omega_2\} &= \omega_2 = 0 \\
\{\Phi_2, \omega_1\} &= -2(\omega_2 + m\sqrt{q_2^2}\Phi_2) = 0 \\
\{\Phi_2, \omega_2\} &= 2p_2^2(q_0q_2) \\
\{\omega_1, \omega_2\} &= q_0^2 - m^2
\end{aligned} \tag{3.12}$$

Conserving the secondary constraints we get

$$\begin{aligned}
\dot{\omega}_1 &= 0 \\
\dot{\omega}_2 &= -q_{0\mu} \left(\frac{m}{\sqrt{q_2^2}} q_2^\mu + q_0^\mu \right) - 2\xi^2 p_2^2 (q_0q_2) = 0
\end{aligned} \tag{3.13}$$

Clearly no more tertiary constraints are obtained. The second condition of (3.13) fixes the multiplier ξ^2

$$\xi^2 = -\frac{1}{2p_2^2(q_0q_2)} \left(\frac{m}{\sqrt{q_2^2}} (q_0q_2) + q_0^2 \right) \tag{3.14}$$

Substituting this in the expression of the total Hamiltonian we get

$$H_T == m\sqrt{q_2^2} + q_{0\mu}q_2^\mu + \xi^1\Phi_1 - \frac{1}{2p_2^2(q_0q_2)} \left(\frac{m}{\sqrt{q_2^2}} (q_0q_2) + q_0^2 \right) \Phi_2 \tag{3.15}$$

It is important to observe that though there are two primary first class constraints only one undetermined multiplier survives in the total Hamiltonian. This shows that effectively there is only one gauge degree of freedom. This feature distinguishes it from a genuine first order theory and has vital implications in the construction of the generator.

We now strongly impose the constraints (3.5). This is possible because the constraints (3.5) merely eliminate the unphysical sector $(q_{0\mu}, p_{0\mu})$ in favour of the physical variables. The constraint (3.10) now read as

$$\omega_1 = p_1q_2 + m\sqrt{q_2^2} \approx 0; \quad \omega_2 = p_1p_2 \approx 0 \tag{3.16}$$

The algebra of the remaining constraints can now be read off from (3.12). We find

$$\{\Phi_2, \omega_2\} = 2p_2^2(p_1q_2); \quad \{\omega_1, \omega_2\} = p_1^2 - m^2 \tag{3.17}$$

If we take

$$\Phi'_2 = (p_1^2 - m^2) \Phi_2 - 2p_2^2 (p_1 q_2) \omega_1 \quad (3.18)$$

then

$$\{\Phi'_2, \omega_2\} = 0 \quad (3.19)$$

If instead of the set of constraints $\Phi_1, \Phi_2, \omega_1, \omega_2$ we take their linear combinations in the form $\Phi_1, \Phi'_2, \omega_1, \omega_2$ we find that only the PB between ω_1 and ω_2 is non-involutive, given by the second equation in (3.17). Clearly Φ_1, Φ'_2 are first class and ω_1, ω_2 are second class.

At this stage an important point is to be noticed. The canonical momentum is set to be equal to the Lagrange multiplier $q_{0\mu}$ when we impose the constraint $\Phi_{1\mu}$ strongly equal to zero. Thus $p_{1\mu}$ is completely arbitrary. It can be space, time or light-like. In the usual relativistic particle model $p_1^2 = m^2$ appears as a constraint of the theory. This is not the case here. In what follows we will assume that $p_1^2 \neq m^2$. In other words we will consider a modified dispersion relation. That such a modified dispersion relation is consistent will be demonstrated by our analysis. The case $p_1^2 = m^2$ will be seen as a singular point in our analysis and will be explored separately in section 3.2.

Now the second class constraints ω_1, ω_2 may be strongly put equal to zero if the PBs $\{A, B\}$ are replaced by DBs $\{A, B\}_D$ ². The non-vanishing DBs are given by³

$$\begin{aligned} \{q_{1\mu}, q_{1\nu}\}_D &= \frac{1}{p_1^2 - m^2} (p_{2\mu} q_{2\nu} - q_{2\mu} p_{2\nu}) \\ \{q_{1\mu}, q_{2\nu}\}_D &= -\frac{q_{2\mu} p_{1\nu}}{p_1^2 - m^2} \\ \{q_{1\mu}, p_{2\nu}\}_D &= \frac{1}{p_1^2 - m^2} \left(\frac{m}{\sqrt{q_2^2}} p_{2\mu} q_{2\nu} + p_{2\mu} p_{1\nu} \right) \end{aligned} \quad (3.20)$$

$$\begin{aligned} \{q_{1\mu}, p_{1\nu}\}_D &= \eta_{\mu\nu} \\ \{q_{2\mu}, p_{2\nu}\}_D &= \eta_{\mu\nu} - \frac{1}{p_1^2 - m^2} \left(p_{1\mu} p_{1\nu} + \frac{m}{\sqrt{q_2^2}} p_{1\mu} q_{2\nu} \right) \end{aligned} \quad (3.21)$$

²Dirac brackets are denoted by $\{, \}_D$ to distinguish them from Poisson brackets which are written as $\{, \}$.

³Note that this computation is valid only when $p_1^2 \neq m^2$. The case $p_1^2 = m^2$ is thus a singular point in our analysis which is to be treated separately.

The structure of the DBs is remarkable. We find that the coordinate algebra $[q_{1\mu}, q_{1\nu}]$ becomes non – commutative. Such non – commutativity is generally known to modify the usual dispersion relation, both effects occurring at Planck scales. Our assumption of the condition $p_1^2 \neq m^2$ is thus consistent and it exhibits the appearance of coordinate noncommutativity in a simple setting including its connection with modified dispersion relation. As shown later in section 4, the treatment of the singular case $p_1^2 = m^2$ naturally leads to a vanishing algebra among the coordinates $q_{1\mu}$

After the substitution of the basic PBs by the DBs there is some simplification of the constraint structure. The second class constraints ω_1 and ω_2 can now be strongly set equal to zero. The constraint Φ'_2 (see equation (3.18)) then reduces to,

$$\Phi'_2 = (p_1^2 - m^2) \Phi_2$$

Let us calculate the DB between the constraints. We find,

$$\{\Phi_1, \Phi'_2\}_D = -2 \left[\omega_2 q_2^2 \left(\omega_2 + \frac{m}{\sqrt{q_2^2}} (p_1 q_2) \right) - p_2^2 (p_1 q_2) \omega_1 \right] = 0 \quad (3.22)$$

since $\omega_1 = \omega_2 = 0$.

We now proceed towards the discussion on gauge symmetry. There are now two first class constraints Φ_1, Φ'_2 , both of which are primary. So the generator of gauge transformation is [33]

$$G = \epsilon^1 \Phi_1 + \epsilon^2 \Phi'_2 \quad (3.23)$$

Had it been a first order theory we would conclude that both ϵ^1, ϵ^2 are independent gauge parameters. However, due to the higher derivative nature we have the additional requirement

$$\frac{d}{d\tau} \delta q_1^\mu = \delta q_2^\mu \quad (3.24)$$

which follows from (2.39). Now

$$\delta q_1^\mu = [q_1^\mu, G] = 2\epsilon^2 p_2^2 q_2^\mu m \sqrt{q_2^2} \quad (3.25)$$

and

$$\delta q_2^\mu = [q_2^\mu, G] = \epsilon^1 q_2^\mu + 2\epsilon^2 q_2^2 (p_1^2 - m^2) p_2^\mu \quad (3.26)$$

Hence using (3.24) we get

$$\epsilon^1 q_2^\mu + 2\epsilon^2 q_2^2 (p_1^2 - m^2) p_2^\mu = \frac{d}{d\tau} \left(2m p_2^2 \sqrt{q_2^2} q_2^\mu \epsilon^2 \right) \quad (3.27)$$

Taking scalar product with $q_{2\mu}$ and using $\Phi_1 = p_2 q_2 \approx 0$, we obtain

$$\epsilon^1 = \frac{q_{2\mu}}{q_2^2} \frac{d}{d\tau} \left(2mp_2^2 \sqrt{q_2^2} \epsilon^2 q_2^\mu \right) \quad (3.28)$$

Clearly, only one parameter ϵ^2 (say) is independent in the gauge generator G . This is also compatible with the observation that there is only one undetermined multiplier in the total Hamiltonian (3.15). The number of independent gauge parameters is thus shown to be less than the number of independent first class primary constraints.

We next show that these findings are consistent with the reparametrization symmetry of the model to which the gauge symmetry is expected to have a one to one correspondence. In the following we will show the mapping between the gauge parameter and the reparametrization parameter.

Consider the following reparametrization

$$\tau \rightarrow \tau + \Lambda \quad (3.29)$$

where Λ is an infinitesimal reparametrization parameter. By direct substitution we can verify that (3.29) is an invariance of (3.1). Now, under this reparametrization x^μ transforms as

$$x'^\mu(\tau) = x^\mu(\tau - \Lambda) \quad (3.30)$$

The variation of x^μ is then

$$\delta x^\mu = x'^\mu(\tau) - x^\mu(\tau) = -\Lambda \dot{x}^\mu \quad (3.31)$$

From equation (3.25) we can write

$$\delta x^\mu = 2\epsilon^2 p_2^2 \dot{x}^\mu m \sqrt{q_2^2} \quad (3.32)$$

where we have used the identification (3.2). Comparing (3.31) and (3.32) we get the desired mapping

$$\Lambda = -2\epsilon^2 p_2^2 m \sqrt{q_2^2} \quad (3.33)$$

Thus in our analysis an exact correspondence between the gauge and reparametrization symmetries is clearly demonstrated.

3.1.1 Behaviour at the singular point ($p_1^2 = m^2$)

In the above Hamiltonian analysis of the relativistic particle model with curvature we have shown that p_1^2 is not constrained by the phase space structure and may be space, time or light-like. In our analysis we have assumed that p_1^2 is not equal to m^2 . This is because the condition $p_1^2 = m^2$ is a singular point and must be treated separately. In the following we will discuss the construction of the gauge generator when the singular limit is assumed. This will also reveal the versatility of our method.

The set of constraints after removal of the unphysical sector $q_0 = p_1$ is now given by,

$$\begin{aligned}\Phi_1 &= p_2 q_2 \approx 0; & \Phi_2 &= p_2^2 q_2^2 + \alpha^2 \approx 0; \\ \omega_1 &= p_1 q_2 + m\sqrt{q_2^2} \approx 0; & \omega_2 &= p_1 p_2 \approx 0\end{aligned}\quad (3.34)$$

Here Φ_1 and Φ_2 are primary while ω_1 and ω_2 are secondary constraints. The constraint algebra shows that Φ_1 and ω_1 are first class while the rest are second class. So the constraint structure in the singular limit is different from that in the general case discussed above. Specifically the appearance of a secondary first class constraint is to be noted. The total Hamiltonian reads as

$$H_T = H_c + \xi^1 \Phi_1 \quad (3.35)$$

where H_c is the canonical Hamiltonian, given by

$$H_c = m\sqrt{q_2^2} + p_1 q_2 \quad (3.36)$$

Note that, as before, the total Hamiltonian consists of one undetermined multiplier indicating one independent gauge degree of freedom.

The construction of the generator of the gauge transformations follow the course outlined in section 2. At first we strongly impose the second class constraints (Φ_2, ω_2) using the Dirac bracket formalism. The non-trivial Dirac brackets among the phase-space variables are

$$\begin{aligned}
\{q_{1\mu}, q_{2\nu}\}_D &= \frac{q_2^2 p_{2\mu} p_{2\nu}}{p_2^2 p_1 q_2} \\
\{q_{1\mu}, p_{2\nu}\}_D &= -\frac{p_{2\mu} q_{2\nu}}{p_1 q_2} \\
\{q_{1\mu}, p_{1\nu}\}_D &= \eta_{\mu\nu} \\
\{q_{2\mu}, p_{2\nu}\}_D &= \eta_{\mu\nu} - \frac{p_{1\mu} q_{2\nu}}{p_1 q_2} \\
\{q_{2\mu}, q_{2\nu}\}_D &= \frac{q_2^2 (p_{1\mu} p_{2\nu} - p_{2\mu} p_{1\nu})}{p_2^2 p_1 q_2}
\end{aligned} \tag{3.37}$$

Note that the coordinate algebra $\{q_{1\mu}, q_{1\nu}\}_D$ becomes commutative as a result of usual dispersion relation. This may be contrasted with the general case ($p_1^2 \neq m^2$) leading to a noncommutative coordinate algebra (first equation in (3.20)). The generator is given by

$$G' = \epsilon^1 \Phi_1 + \epsilon^2 \omega_1 \tag{3.38}$$

Due to the presence of a secondary first class constraint (ω_1), ϵ^1 and ϵ^2 are not independent. There is a restriction [32]

$$\frac{d\epsilon^2}{d\tau} - \sum_{a=1,2} \epsilon^a (V_a^2 + \xi^1 C_{1a}^2) = 0 \tag{3.39}$$

The other restriction(3.24) follows from the higher derivative nature . Interestingly, both the restrictions lead to the same condition

$$\epsilon^1 = \dot{\epsilon}^2 + \xi^1 \epsilon^2 \tag{3.40}$$

To proceed further we require to express the Lagrange multiplier ξ^1 in terms of the phase space variables. To this end we calculate \dot{q}_2^μ as

$$\dot{q}_2^\mu = \{q_2^\mu, H_T\}_D \tag{3.41}$$

where H_T is the total Hamiltonian given by equation (3.35). Using the basic brackets (3.37) we get

$$\begin{aligned}
\dot{q}_2^\mu &= \xi^1 \left[q_2^\mu - \frac{p_1^\mu}{p_1^2 - m^2} \left(p_1 q_2 + m \sqrt{q_2^2} \right) \right] - \frac{q_2^2}{p_2^2 p_1 q_2} \left(\frac{m}{\sqrt{q_2^2}} (p_1 q_2) + p_1^2 \right) \\
&\quad \left[p_2^\mu - \frac{p_1^\mu}{p_1^2 - m^2} \left(p_1 p_2 + \frac{m}{\sqrt{q_2^2}} p_2 q_2 \right) \right]
\end{aligned} \tag{3.42}$$

Implementing the constraints (3.34) and simplifying, yields,

$$\dot{q}_2^\mu = \xi^1 q_2^\mu - (p_1^2 - m^2) \frac{q_2^2 p_2^\mu}{p_2^2 p_1 q_2}, \quad (3.43)$$

which immediately gives, on contraction with $q_{2\mu}$,

$$\xi^1 = \frac{1}{2q_2^2} \frac{d}{d\tau} (q_2^2) \quad (3.44)$$

Using (3.40, 3.44) we can express one of the gauge parameters appearing in G' (3.38) in terms of the other. We observe that in spite of the modified constraint structure corresponding to the singular point, the gauge generator may be consistently constructed by our general method.

3.2 A consistency check

An important relation to check the consistency of our scheme is given by [32]

$$\delta\xi^1 = \frac{d\epsilon^1}{dt} - \sum_{a=1,2} \epsilon^a (V_a^1 + \xi^1 C_{1a}^1) \quad (3.45)$$

where the coefficients V_a^b and C_{ab}^c are defined by (2.19, 2.20). Here both V_a^b and C_{ab}^c vanish.

Now we can independently calculate $\delta\xi^1$ for our theory and an exact agreement with (3.45) will be demonstrated.

Taking the gauge variation on both sides of (3.44) and substituting δq_2^μ from (3.26) we arrive at,

$$\begin{aligned} \delta\xi^1 = & - \frac{1}{(q_2^2)^2} q_{2\mu} (\epsilon^1 q_2^\mu + 2\epsilon^2 q_2^2 (p_1^2 - m^2) p_2^\mu) \frac{d}{d\tau} q_2^2 \\ & + \frac{1}{q_2^2} \frac{d}{d\tau} [q_{2\mu} (\epsilon^1 q_2^\mu + 2\epsilon^2 q_2^2 (p_1^2 - m^2) p_2^\mu)] \end{aligned} \quad (3.46)$$

Imposing the constraints (3.6, 3.16) and simplifying the ensuing algebra immediately reproduces(3.45).This completes our consistency check.

3.3 The rigid relativistic particle model

The massless version of the model known as ‘rigid relativistic particle’, is obtained by setting $m = 0$ in (3.1). It presents some unique features. We perform a detailed Hamiltonian analysis which is quite distinct from the earlier(massive) model due to a modified symplectic structure. It is interesting to point out in this case that we will be able to find out an extra symmetry apart from the expected diffeomorphism symmetry. This is the W_3 -symmetry [45].

3.3.1 Hamiltonian analysis

The relativistic point particle theory with rigidity only has the action [29,31]

$$S = \alpha \int \frac{((\dot{x}\ddot{x})^2 - \dot{x}^2\ddot{x}^2)^{\frac{1}{2}}}{\dot{x}^2} d\tau \quad (3.47)$$

We introduce the new coordinates

$$q_1^\mu = x^\mu \quad q_2^\mu = \dot{x}^\mu \quad (3.48)$$

The Lagrangian in these coordinates has a first order form given by

$$L = \alpha \frac{((q_2\dot{q}_2)^2 - q_2^2\dot{q}_2^2)^{\frac{1}{2}}}{q_2^2} + q_0^\mu (\dot{q}_{1\mu} - q_{2\mu}) \quad (3.49)$$

where q_0^μ are the Lagrange multipliers that enforce the constraints

$$\dot{q}_{1\mu} - q_{2\mu} = 0 \quad (3.50)$$

Let $p_{0\mu}$, $p_{1\mu}$ and $p_{2\mu}$ be the canonical momenta conjugate to $q_{0\mu}$, $q_{1\mu}$ and $q_{2\mu}$ respectively. Then

$$\begin{aligned} p_{0\mu} &= \frac{\partial L}{\partial \dot{q}_0^\mu} = 0 \\ p_{1\mu} &= q_{0\mu} \\ p_{2\mu} &= \frac{\alpha}{q_2^2 \sqrt{g}} l_\mu \end{aligned} \quad (3.51)$$

where,

$$\begin{aligned} g &= ((q_2\dot{q}_2)^2 - q_2^2\dot{q}_2^2) \\ l_\mu &= (q_2\dot{q}_2)q_{2\mu} - q_2^2\dot{q}_{2\mu} \end{aligned} \quad (3.52)$$

We immediately get the following primary constraints

$$\begin{aligned}\Phi_{0\mu} &= p_{0\mu} \approx 0 \\ \Phi_{1\mu} &= p_{1\mu} - q_{0\mu} \approx 0\end{aligned}\tag{3.53}$$

and

$$\begin{aligned}\Phi_1 &= p_2 q_2 \approx 0 \\ \Phi_2 &= p_2^2 q_2^2 + \alpha^2 \approx 0\end{aligned}\tag{3.54}$$

The first set of constraints (3.53) are an outcome of our extension of the original Lagrangian. The canonical Hamiltonian following from the usual definition is given by

$$H_C = q_{0\mu} q_2^\mu\tag{3.55}$$

The total Hamiltonian is

$$H_T = H_C + u_{0\mu} \Phi_{0\mu}^\mu + u_{1\mu} \Phi_{1\mu}^\mu + \lambda^1 \Phi_1 + \lambda^2 \Phi_2\tag{3.56}$$

where $u_{0\mu}$, $u_{1\mu}$, λ^1 and λ^2 are as yet undetermined multipliers. Now, conserving the primary constraints we get

$$\begin{aligned}\dot{\Phi}_{0\mu} &= \{\Phi_{0\mu}, H_T\} = -q_{2\mu} + u_{1\mu} \approx 0 \\ \dot{\Phi}_{1\mu} &= \{\Phi_{1\mu}, H_T\} = -u_{0\mu} \approx 0 \\ \dot{\Phi}_1 &= \{\Phi_1, H_T\} = -q_0 q_2 \approx 0 \\ \dot{\Phi}_2 &= \{\Phi_2, H_T\} = -2(q_0 p_2) q_2^2 \approx 0\end{aligned}\tag{3.57}$$

We find that the multipliers $u_{0\mu}$ and $u_{1\mu}$ are fixed

$$u_{0\mu} = 0 \text{ and } u_{1\mu} = q_{2\mu}\tag{3.58}$$

while new secondary constraints are obtained

$$\begin{aligned}\omega_1 &= q_0 q_2 \approx 0 \\ \omega_2 &= q_0 p_2 \approx 0\end{aligned}\tag{3.59}$$

The last constraint in (3.57) simplifies to ω_2 since $q_2^2 \neq 0$ as may be observed from the structure of the Lagrangian (3.49). The total Hamiltonian now becomes

$$H_T = q_{0\mu} q_2^\mu + q_{2\mu} \Phi_{1\mu}^\mu + \lambda^1 \Phi_1 + \lambda^2 \Phi_2\tag{3.60}$$

Preserving the constraints 3.59 we get $\dot{\omega}_1 \approx 0$; $\dot{\omega}_2 \approx -q_0^2$ thereby yielding a tertiary constraint $\Phi_3 = q_0^2 \approx 0$. This terminates the iterative process of

obtaining constraints. The complete set of constraints is now given by,

$$\begin{aligned}
\Phi_{0\mu} &= p_{0\mu} \approx 0 \\
\Phi_{1\mu} &= p_{1\mu} - q_{0\mu} \approx 0 \\
\Phi_1 &= p_2 q_2 \approx 0 \\
\Phi_2 &= p_2^2 q_2^2 + \alpha^2 \approx 0 \\
\omega_1 &= q_0 q_2 \approx 0 \\
\omega_2 &= q_0 p_2 \approx 0 \\
\Phi_3 &= q_0^2 \approx 0
\end{aligned} \tag{3.61}$$

PBs among the constraints are given by

$$\begin{aligned}
\{\Phi_{0\mu}, \Phi_{1\nu}\} &= \eta_{\mu\nu} \\
\{\Phi_{0\mu}, \omega_1\} &= -q_{2\mu} \\
\{\Phi_{0\mu}, \omega_2\} &= -p_{2\mu} \\
\{\Phi_{0\mu}, \Phi_3\} &= -2q_{0\mu} \\
\{\Phi_1, \omega_1\} &= -\omega_1 = 0 \\
\{\Phi_1, \omega_2\} &= \omega_2 = 0 \\
\{\Phi_1, \Phi_2\} &= 0 \\
\{\Phi_2, \omega_1\} &= -2q_2^2 \omega_2 = 0 \\
\{\Phi_2, \omega_2\} &= 2p_2^2 (q_0 q_2) = 0 \\
\{\omega_1, \omega_2\} &= q_0^2 = 0
\end{aligned} \tag{3.62}$$

Apparently, $\Phi_{0\mu}$, $\Phi_{1\mu}$, ω_1 , ω_2 and Φ_3 are second class. However we may substitute Φ_3 by Φ'_3 where

$$\Phi'_3 = \Phi_3 - 2q'_0 \Phi_{1\nu} \tag{3.63}$$

One can easily verify that Φ'_3 commutes with all the constraints. The set of constraints are taken to be $\Phi_{0\mu}$, $\Phi_{1\mu}$, Φ_1 , Φ_2 , ω_1 , ω_2 and Φ'_3 which are may be classified in table 1.

3.3.2 Gauge symmetries and the emergence of the W_3 algebra

In the above we have considered the hamiltonian formulation of the model (3.47) in the first order approach. The expression of the total Hamiltonian

Table 3.1: Classification of Constraints

	First class	Second class
Primary	Φ_1, Φ_2	$\Phi_{0\mu}, \Phi_{1\nu}$
Secondary	Φ'_3	ω_1, ω_2

(3.60) suggests two independent gauge symmetries of the model. In the following we will construct the gauge generator and interpret the different gauge symmetries physically.

The variables $q_{0\mu}$ and their associated momenta $p_{0\mu}$ comprise the unphysical sector of the phase space. This is characterised by the second class pair $\Phi_{0\mu}$ and $\Phi_{1\mu}$. To find the gauge symmetries we have to eliminate these constraints. Considering their unphysical nature it will be appropriate to work in the reduced phase by putting $\Phi_{0\mu}$ and $\Phi_{1\mu}$ equal to zero. The set of remaining constraints in the reduced phase space become

$$\begin{aligned}
\Phi_1 &= p_2 q_2 \approx 0 \\
\Phi_2 &= p_2^2 q_2^2 + \alpha^2 \approx 0 \\
\omega_1 &= p_1 q_2 \approx 0 \\
\omega_2 &= p_1 p_2 \approx 0 \\
\Phi_3 &= p_1^2 \approx 0
\end{aligned} \tag{3.64}$$

Inspection of the algebra (3.62) shows that in the reduced phase space $\Phi_1, \Phi_2, \omega_1, \omega_2$ and Φ'_3 form a first class set. In what follows it will be advantageous to rename the constraints as

$$\begin{aligned}
\Omega_1 = \Phi_1 &= p_2 q_2 \approx 0 \\
\Omega_2 = \Phi_2 &= p_2^2 q_2^2 + \alpha^2 \approx 0 \\
\Omega_3 = \omega_1 &= p_1 q_2 \approx 0 \\
\Omega_4 = \omega_2 &= p_1 p_2 \approx 0 \\
\Omega_5 = \Phi_3 &= p_1^2 \approx 0
\end{aligned} \tag{3.65}$$

Also the canonical Hamiltonian is obtained from (3.55) as

$$H_c = \Omega_3 \tag{3.66}$$

The corresponding total Hamiltonian is

$$H_T = H_C + \lambda^1 \Phi_1 + \lambda^2 \Phi_2 \tag{3.67}$$

As done previously, the gauge generator is written as a combination of all the first class constraints,

$$G = \sum_{a=1}^5 \epsilon^a \Omega_a. \quad (3.68)$$

However, due to the presence of secondary first-class constraints, the parameters of gauge transformation (ϵ_a) are not independent. The independent parameters will be isolated by using 2.18. The first step in this direction is to calculate the structure functions V_a^b and C_{ab}^c using their definitions (2.19, 2.20). A straightforward calculation gives the following nonzero values:

$$\begin{aligned} V_1^3 &= 1, & V_2^4 &= 2q_2^2 \\ V_4^5 &= 1 \end{aligned} \quad (3.69)$$

and

$$\begin{aligned} C_{13}^3 &= -1, & C_{14}^4 &= 1 \\ C_{23}^4 &= -2q_2^2, & C_{24}^3 &= 2p_2^2 \\ C_{34}^5 &= 1 \end{aligned} \quad (3.70)$$

Using these (3.69, 3.70) in the master equation (2.18) we arrive at the following equations

$$\begin{aligned} \dot{\epsilon}^3 - \epsilon^1 + \lambda^1 \epsilon^3 - 2\lambda^2 \epsilon^4 p_2^2 &= 0 \\ \dot{\epsilon}^4 - 2q_2^2 \epsilon^2 - \lambda^1 \epsilon^4 + 2\lambda^2 \epsilon^3 q_2^2 &= 0 \\ \dot{\epsilon}^5 &= \epsilon^4 \end{aligned} \quad (3.71)$$

Certain points are immediately apparent from the above equations. Out of the five gauge parameters ϵ^a three parameters may be expressed in terms of the remaining two using the equations (3.71). It is most convenient to take ϵ^3 and ϵ^5 as independent.

To proceed further we require to work out the Lagrange multipliers in (3.67). For that purpose we first compute $\dot{q}_{2\mu} = \{q_{2\mu}, H_T\}$. Using the expression for total Hamiltonian given in (3.67) we get

$$\dot{q}_{2\mu} = \lambda^1 q_{2\mu} + 2\lambda^2 q_2^2 p_{2\mu} \quad (3.72)$$

Now scalar multiplying the above equation (3.72) by q_2^μ and p_2^μ we obtain respectively solutions for λ^1 and λ^2 as

$$\begin{aligned} \lambda^1 &= \frac{q_2 \dot{q}_2}{q_2^2} \\ \lambda^2 &= \frac{\alpha \sqrt{g}}{2p_2^2 q_2^4} \end{aligned} \quad (3.73)$$

Substituting these results for λ^1 , λ^2 and $\epsilon^4 = \dot{\epsilon}^5$ in the first two equations of (3.71) one can express

$$\begin{aligned}\epsilon^1 &= \dot{\epsilon}^3 + \frac{q_2 \dot{q}_2}{q_2^2} \epsilon^3 - \frac{\alpha \sqrt{g}}{q_2^4} \dot{\epsilon}^5 \\ \epsilon^2 &= \frac{1}{2q_2^2} \left(\ddot{\epsilon}^5 - \frac{q_2 \dot{q}_2}{q_2^2} \dot{\epsilon}^5 + \frac{\alpha \sqrt{g}}{p_2^2 q_2^2} \epsilon^3 \right)\end{aligned}\quad (3.74)$$

Note that the theory under investigation is a higher derivative theory. So the gauge transformations are additionally subject to the condition (2.39). After simplification this condition reduces to

$$2q_2 \dot{p}_1 \dot{\epsilon}^5 = \left(q_2 \dot{p}_2 + \frac{\alpha \sqrt{g}}{q_2^2} \right) \quad (3.75)$$

Using the constraints of the theory this condition reduces to a tautological statement $0 = 0$. So, for this particular model, (2.39) does not impose any new conditions on the gauge parameters. We thus find that there are two independent gauge parameters in the expression of the gauge generator G . Taking ϵ^3 and $\dot{\epsilon}^5$ to be independent, the expression for the gauge generator becomes (3.68)

$$\begin{aligned}G &= \left(\dot{\epsilon}^3 + \frac{q_2 \dot{q}_2}{q_2^2} \epsilon^3 - \frac{\alpha \sqrt{g}}{q_2^4} \dot{\epsilon}^5 \right) \Omega_1 + \\ &\quad \frac{1}{2q_2^2} \left(\ddot{\epsilon}^5 - \frac{q_2 \dot{q}_2}{q_2^2} \dot{\epsilon}^5 + \frac{\alpha \sqrt{g}}{p_2^2 q_2^2} \epsilon^3 \right) \Omega_2 + \epsilon^3 \Omega_3 + \dot{\epsilon}^5 \Omega_4 + \epsilon^5 \Omega_5\end{aligned}\quad (3.76)$$

that there are only two independent parameters is consistent with the fact that there were two independent Lagrange multipliers in the expression of the total hamiltonian (3.67). Note, however, the distinction from the massive model discussed in the previous section. There we found only one independent gauge degree of freedom which was shown to have a one to one correspondence with the diffeomorphism invariance of the model. Clearly, the rigid relativistic particle is endowed with more general symmetries as is indicated by its gauge generator.

In order to unravel the meaning of the additional gauge symmetry we calculate the gauge variations of the dynamical variables, defined as $\delta q = \{q, G\}$.

These are given by,

$$\begin{aligned}
\delta q_1^\mu &= \epsilon^3 q_2^\mu + \dot{\epsilon}^5 p_2^\mu + 2\epsilon^5 p_1^\mu \\
\delta q_2^\mu &= \left(\dot{\epsilon}^3 + \frac{q_2 \dot{q}_2}{q_2^2} \epsilon^3 - \frac{\alpha \sqrt{g}}{q_2^4} \dot{\epsilon}^5 \right) q_2^\mu + \left(\ddot{\epsilon}^5 - \frac{q_2 \dot{q}_2}{q_2^2} \dot{\epsilon}^5 + \frac{\alpha \sqrt{g}}{p_2^2 q_2^2} \epsilon^3 \right) p_2^\mu + \dot{\epsilon}^5 p_1^\mu \\
\delta p_1^\mu &= 0 \\
\delta p_2^\mu &= - \left(\dot{\epsilon}^3 + \frac{q_2 \dot{q}_2}{q_2^2} \epsilon^3 - \frac{\alpha \sqrt{g}}{q_2^4} \dot{\epsilon}^5 \right) p_2^\mu \\
&\quad - \frac{p_2^2}{q_2^2} \left(\ddot{\epsilon}^5 - \frac{q_2 \dot{q}_2}{q_2^2} \dot{\epsilon}^5 + \frac{\alpha \sqrt{g}}{p_2^2 q_2^2} \epsilon^3 \right) q_2^\mu - \epsilon^3 p_1^\mu
\end{aligned} \tag{3.77}$$

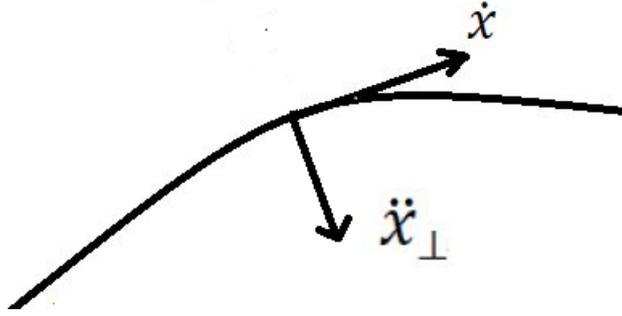


Figure 3.1: orthogonal frame attached with particle world trajectory

Now it would be customary to identify the geometrical origin of these independent gauge transformations. It is well known that the most general deformation of the rigid particle trajectory may be resolved as [46]

$$\delta_{\beta, \eta} x = \beta(t) \dot{x} + \eta(t) \ddot{x}_\perp \tag{3.78}$$

where $\ddot{x}_\perp^\mu = \frac{l^\mu}{x^2}$ is orthogonal to the tangent space mapped by \dot{x} (see figure.1). The coefficients β , η are respectively the diffeo-morphism and w-morphism parameters. Now we rewrite the first equation of (3.77) as

$$\delta q_1^\mu = \epsilon^3 \dot{x}^\mu + \dot{\epsilon}^5 \frac{\alpha}{\sqrt{g}} \ddot{x}_\perp^\mu + 2\epsilon^5 p_1^\mu \tag{3.79}$$

As we have noted earlier, the gauge variations (3.77) contain two independent gauge parameters ϵ^3 and ϵ^5 . Let us first assume

$$\epsilon^3 \neq 0 \quad \text{and} \quad \epsilon^5 = 0 \tag{3.80}$$

Substituting this condition in (3.79) we get

$$\delta q_1^\mu = \epsilon^3 \dot{x}^\mu \tag{3.81}$$

Comparing the above with (3.78) we can easily see that the gauge generator subject to the limiting condition (3.80) generates diffeomorphism (with $\epsilon^3 = \beta$) and the corresponding gauge symmetry is identified with diffeomorphism invariance. This identification may be confirmed by working out the variations of other phase space variables under reparametrization and comparing them with the gauge variation generated by G , subject to the same condition (3.80).

It is now clear that the gauge generator subject to the other extreme condition

$$\epsilon^3 = 0 \quad \text{and} \quad \epsilon^5 \neq 0 \quad (3.82)$$

generates symmetry transformations other than diffeos. Substituting (3.82) in (3.79) we now get

$$\delta q_1^\mu = \epsilon^5 \frac{\alpha}{\sqrt{g}} \ddot{x}_\perp^\mu + 2\epsilon^5 p_1^\mu \quad (3.83)$$

Looking back at the second equation of (3.51) we find that the linear term containing p_1^μ in the variation (3.83) can be neglected, as p_1^μ is a Lagrange multiplier. Comparing with (3.78) we can then associate (3.83) with w - transformation, by identifying $\eta = \epsilon^5 \frac{\alpha}{\sqrt{g}}$. Thus the additional symmetry corresponding to (3.82) is nothing but W - symmetry. It is now straightforward to show that the two different symmetries together satisfy the W_3 algebra. Let us denote the transformations of category 1 (diffeomorphisms) by the superscript 'D' and the category 2 transformations by 'W'. Detailed calculations on all the phase-space variables show that

$$\begin{aligned} \left\{ \delta_{\epsilon_3^3}^{(D)}, \delta_{\epsilon_3^3}^{(D)} \right\} &= \delta_{\epsilon_3^3}^{(D)}; \quad \text{with} \quad \epsilon^3 = \epsilon_1^3 \epsilon_2^3 - \epsilon_1^3 \epsilon_2^3 \\ \left\{ \delta_{\epsilon_3^3}^{(D)}, \delta_{\epsilon_5^5}^{(W)} \right\} &= \delta_{\epsilon_5^5}^{(W)}; \quad \text{with} \quad \epsilon'^5 = -\epsilon^3 \epsilon^5 \\ \left\{ \delta_{\epsilon_1^5}^{(W)}, \delta_{\epsilon_2^5}^{(W)} \right\} &= \delta_{\epsilon_5^5}^{(W)}; \quad \text{with} \quad \epsilon^5 = \frac{p_2^2}{q_2^2} (\epsilon_2^5 \epsilon_1^5 - \epsilon_2^5 \epsilon_1^5) \end{aligned}$$

which is nothing but the W_3 algebra.

3.4 (Anti-)BRST symmetries and W_3 -algebra

In this section we construct the nilpotent BRST and anti-BRST symmetries for the theory. For this purpose we need to fix a gauge before the quantization of the theory as the theory is gauge invariant and therefore has some redundant degrees of freedom. The general gauge condition in this case is chosen as:

$$F_1[f(q)] = 0, \quad (3.84)$$

where $f(q)$ is a general function of all the generic variables q . Some explicit examples of gauge conditions corresponding to relativistic particle models are [31].

$$q_1^0 - \tau = 0, \quad q_2^0 - 1 = 0, \quad p_2^0 = 0, \quad q_2^2 = 0. \quad (3.85)$$

The general gauge condition (6.46) can be incorporated at a quantum level by adding the appropriate gauge-fixing term to classical action.

The linearised gauge-fixing term using Nakanishi-Lautrup auxiliary variable $B(q)$ is given by

$$S_{gf} = \int d\tau \left[\frac{1}{2} B^2 + B F_1[f(q)] \right]. \quad (3.86)$$

To complete the effective theory we need a further Faddeev-Popov ghost term in the action. The ghost term in this case is constructed as

$$\begin{aligned} S_{gh} &= \int d\tau [\bar{c} s F_1[f(q)]], \\ &= - \int d\tau [c \bar{s} F_1[f(q)]], \end{aligned} \quad (3.87)$$

where c and \bar{c} are ghost and anti-ghost variables. Now the effective action can be written as

$$S_{eff} = S + S_{gf} + S_{gh}. \quad (3.88)$$

The source free generating functional for this theory is defined as

$$Z[0] = \int \mathcal{D}q e^{iS_{eff}}, \quad (3.89)$$

where $\mathcal{D}q$ is the path integral measure. The nilpotent BRST symmetry of the effective action in the case of relativistic particle model with curvature is defined by replacing the infinitesimal reparametrisation parameter (Λ) to ghost variable c in the gauge transformation given in equation (3.25) as

$$\begin{aligned} s^D q_1^\mu &= -c q_2^\mu, & s^D q_2^\mu &= -\dot{c} q_2^\mu - c \dot{q}_2^\mu, \\ s^D c &= 0, & s^D \bar{c} &= B, & s^D B &= 0, \end{aligned} \quad (3.90)$$

where c , \bar{c} and B are ghost, anti-ghost and auxiliary variables respectively for relativistic particle model with curvature. This BRST transformation, corresponding to gauge symmetry identified with the diffeomorphism invariance, leaves both the effective action as well as generating functional, invariant. Similarly, we construct the anti-BRST symmetry transformation, where the roles of ghost and anti-ghosts are interchanged with some coefficients, as

$$\begin{aligned} \bar{s}^D q_1^\mu &= -\bar{c} q_2^\mu, & \bar{s}^D q_2^\mu &= -\dot{\bar{c}} q_2^\mu - \bar{c} \dot{q}_2^\mu, \\ \bar{s}^D \bar{c} &= 0, & \bar{s}^D c &= -B, & \bar{s}^D B &= 0. \end{aligned} \quad (3.91)$$

These transformations are nilpotent and absolutely anticommuting in nature i.e.

$$(s^D)^2 = 0, \quad (\bar{s}^D)^2 = 0, \quad s^D \bar{s}^D + \bar{s}^D s^D = 0. \quad (3.92)$$

The above (anti-)BRST transformations are valid for both the models. On the other hand, the nilpotent BRST and anti-BRST symmetry transformations, identified with W -symmetry (with $\epsilon^3 = 0$) in (3.77), for relativistic massless particle model with rigidity only, are constructed as

$$\begin{aligned} s^W q_1^\mu &= \dot{\eta} p_2^\mu + 2\eta p_1^\mu, \\ s^W q_2^\mu &= -\frac{\alpha\sqrt{g}}{q_2^4} \dot{\eta} q_2^\mu + \left(\ddot{\eta} - \frac{q_2 \dot{q}_2}{q_2^2} \dot{\eta} \right) p_2^\mu + \dot{\eta} p_1^\mu, \\ s^W p_1^\mu &= 0, \quad s^W p_2^\mu = \frac{\alpha\sqrt{g}}{q_2^4} \dot{\eta} p_2^\mu - \frac{p_2^2}{q_2^2} \left(\ddot{\eta} - \frac{q_2 \dot{q}_2}{q_2^2} \dot{\eta} \right) q_2^\mu, \\ s^W \eta &= 0, \quad s^W \bar{\eta} = B, \quad s^W B = 0, \end{aligned} \quad (3.93)$$

and

$$\begin{aligned} \bar{s}^W q_1^\mu &= \dot{\bar{\eta}} p_2^\mu + 2\bar{\eta} p_1^\mu, \\ \bar{s}^W q_2^\mu &= -\frac{\alpha\sqrt{g}}{q_2^4} \dot{\bar{\eta}} q_2^\mu + \left(\ddot{\bar{\eta}} - \frac{q_2 \dot{q}_2}{q_2^2} \dot{\bar{\eta}} \right) p_2^\mu + \dot{\bar{\eta}} p_1^\mu, \\ \bar{s}^W p_1^\mu &= 0, \quad \bar{s}^W p_2^\mu = \frac{\alpha\sqrt{g}}{q_2^4} \dot{\bar{\eta}} p_2^\mu - \frac{p_2^2}{q_2^2} \left(\ddot{\bar{\eta}} - \frac{q_2 \dot{q}_2}{q_2^2} \dot{\bar{\eta}} \right) q_2^\mu, \\ \bar{s}^W \bar{\eta} &= 0, \quad \bar{s}^W \eta = -B, \quad \bar{s}^W B = 0, \end{aligned} \quad (3.94)$$

where $\eta, \bar{\eta}$ and B are ghost, anti-ghost and auxiliary variables, respectively, for relativistic massless particle model with rigidity.

Here we observe interestingly that the BRST symmetry transformations of all variables (excluding the anti-ghost variable) given in equations (3.90) and (3.93) also satisfy the W_3 -algebra as

$$\begin{aligned} [s_{c_1}^D, s_{c_2}^D] &= s_{c_3}^D; \quad \text{with} \quad c_3 = c_2 \dot{c}_1 - \dot{c}_2 c_1 \\ [s_c^D, s_\eta^W] &= s_{\eta'}^W; \quad \text{with} \quad \eta' = \dot{\eta} c \\ [s_{\eta_1}^W, s_{\eta_2}^W] &= s_{\eta_3}^W; \quad \text{with} \quad \eta_3 = \frac{p_2^2}{q_2^2} (\eta_2 \dot{\eta}_1 - \eta_1 \dot{\eta}_2), \end{aligned}$$

and the anti-BRST symmetry transformations of all variables (excluding ghost variable) given in equations (3.91) and (3.94) also satisfy the W_3 -algebra as

$$\begin{aligned} [\bar{s}_{\bar{c}_1}^D, \bar{s}_{\bar{c}_2}^D] &= \bar{s}_{\bar{c}_3}^D; \quad \text{with} \quad \bar{c}_3 = \bar{c}_2 \dot{\bar{c}}_1 - \dot{\bar{c}}_2 \bar{c}_1 \\ [\bar{s}_{\bar{c}}^D, \bar{s}_{\bar{\eta}}^W] &= \bar{s}_{\bar{\eta}'}^W; \quad \text{with} \quad \bar{\eta}' = \dot{\bar{\eta}} \bar{c} \\ [\bar{s}_{\bar{\eta}_1}^W, \bar{s}_{\bar{\eta}_2}^W] &= \bar{s}_{\bar{\eta}_3}^W; \quad \text{with} \quad \bar{\eta}_3 = \frac{p_2^2}{q_2^2} (\bar{\eta}_2 \dot{\bar{\eta}}_1 - \bar{\eta}_1 \dot{\bar{\eta}}_2). \end{aligned}$$

This completes our analysis of the connection between the (anti-)BRST symmetries and W_3 -algebra.

3.5 FCBRST formulation for higher derivative theory

In this section we investigate the finite coordinate-dependent BRST (FCBRST) formulation for general higher derivative theory. To do so, we first define the infinitesimal BRST symmetry transformation with Grassmannian constant parameter $\delta\rho$ as

$$\delta_b q = s q \delta\rho, \quad (3.95)$$

where $s q$ is the BRST variation of generic variables q in the higher derivative theories. The properties of the usual BRST transformation in equation (3.95) do not depend on whether the parameter $\delta\rho$ is (i) finite or infinitesimal, (ii) variable-dependent or not, as long as it is anticommuting and global in nature. These observations give us a freedom to generalize the BRST transformation by making the parameter $\delta\rho$ finite and coordinate-dependent without affecting its properties. We call such generalized BRST transformation in quantum mechanical systems as FCBRST transformation. In the field theory such generalization is known as FFBRST transformation [53]. Here we adopt a similar technique to generalize the BRST transformation in quantum mechanical theory. We start by making the infinitesimal parameter coordinate-dependent with introduction of an arbitrary parameter κ ($0 \leq \kappa \leq 1$). We allow the generalized coordinates, $q(\kappa)$, to depend on κ in such a way that $q(\kappa = 0) = q$ and $q(\kappa = 1) = q'$, the transformed coordinate.

The usual infinitesimal BRST transformation, thus can be written generically as

$$dq(\kappa) = s[q]\Theta'[q(\kappa)]d\kappa, \quad (3.96)$$

where the $\Theta'[q(\kappa)]d\kappa$ is the infinitesimal but coordinate-dependent parameter. The FCBRST transformation with the finite coordinate-dependent parameter then can be constructed by integrating such infinitesimal transformation from $\kappa = 0$ to $\kappa = 1$, to obtain [53]

$$q' \equiv q(\kappa = 1) = q(\kappa = 0) + s(q)\Theta[q], \quad (3.97)$$

where

$$\Theta[q] = \int_0^1 d\kappa' \Theta'[q(\kappa')], \quad (3.98)$$

is the finite coordinate-dependent parameter.

Such a generalized BRST transformation with finite coordinate-dependent parameter is the symmetry of the effective action in equation (3.88). However, the path integral measure in equation (3.89) is not invariant under such transformation as the BRST parameter is finite in nature. The Jacobian of the path integral measure for such transformations is then evaluated for some particular choices of the finite coordinate-dependent parameter, $\Theta[q(x)]$, as

$$\mathcal{D}q' = J(\kappa)\mathcal{D}q(\kappa). \quad (3.99)$$

The Jacobian, $J(\kappa)$ can be replaced (within the functional integral) as

$$J(\kappa) \rightarrow \exp[iS_1[q(\kappa)]], \quad (3.100)$$

iff the following condition is satisfied [53]

$$\int \mathcal{D}q \left[\frac{1}{J} \frac{dJ}{d\kappa} - i \frac{dS_1[q(x, \kappa)]}{d\kappa} \right] \exp[i(S_{eff} + S_1)] = 0, \quad (3.101)$$

where $S_1[q]$ is local functional of variables such that at $\kappa = 0$ it must vanish.

The infinitesimal change in the $J(\kappa)$ is written as [53],

$$\frac{1}{J} \frac{dJ}{d\kappa} = - \int d\tau \left[\pm sq(\kappa) \frac{\partial \Theta'[q(\kappa)]}{\partial q(\kappa)} \right], \quad (3.102)$$

where \pm sign refers to whether q is a bosonic or a fermionic variable.

Thus, the FCBRSST transformation with appropriate Θ , changes the effective action S_{eff} to a new effective action $S_{eff} + S_1(\kappa = 1)$ within the functional integration.

3.6 Connecting different gauges in relativistic particle models

Here we will exploit the general FCBRSST formulation developed in the previous section to connect the path integral of relativistic particle models with different gauge conditions. The FCBRSST transformations (f_b) for the relativistic particle model with curvature are constructed as follows:

$$\begin{aligned} f_b q_1^\mu &= -c q_2^\mu \Theta[q], & f_b q_2^\mu &= (-\dot{c} q_2^\mu - c \dot{q}_2^\mu) \Theta[q], \\ f_b c &= 0, & f_b \bar{c} &= B \Theta[q], & f_b B &= 0, \end{aligned} \quad (3.103)$$

where $\Theta[q]$ is an arbitrary finite coordinate-dependent parameter. Now, we show how two different gauges (say $F_1(q) = 0$ and $F_2(q) = 0$) in the relativistic particle model may be connected by such transformations. For this purpose, let us choose the following infinitesimal coordinate dependent parameter (through equation (3.98))

$$\Theta'[q] = -i \int d\tau \bar{c}(F_1 - F_2). \quad (3.104)$$

Let us first calculate the infinitesimal change in the Jacobian $J(\kappa)$ for above $\Theta'[q]$ using the relation (3.102) as

$$\begin{aligned} \frac{1}{J} \frac{dJ}{d\kappa} &= i \int d\tau [-B(F_1 - F_2) + s(F_1 - F_2)\bar{c}], \\ &= -i \int d\tau [B(F_1 - F_2) + \bar{c} s(F_1 - F_2)]. \end{aligned} \quad (3.105)$$

To express the Jacobian as e^{iS_1} [53], we take the ansatz,

$$S_1[\kappa] = \int d\tau [\zeta_1(\kappa)BF_1 + \zeta_2(\kappa)BF_2 + \zeta_3(\kappa)\bar{c} sF_1 + \zeta_4(\kappa)\bar{c} sF_2], \quad (3.106)$$

where $\zeta_i(\kappa)$ ($i = 1, \dots, 4$) are constant parameters satisfying the boundary conditions

$$\zeta_i(\kappa = 0) = 0. \quad (3.107)$$

To satisfy the crucial condition (3.101), we calculate the infinitesimal change in S_1 with respect to κ using the relation (3.96) as

$$\begin{aligned} \frac{dS_1[q, \kappa]}{d\kappa} &= \int d\tau [\zeta'_1 BF_1 + \zeta'_2 BF_2 + \zeta'_3 \bar{c} sF_1 + \zeta'_4 \bar{c} sF_2 \\ &+ (\zeta_1 - \zeta_2)B(sF_1)\Theta' + (\zeta_2 - \zeta_4)B(sF_2)\Theta'], \end{aligned} \quad (3.108)$$

where prime denotes the differentiation with respect to κ . Exploiting equations (3.105) and (3.108), the condition (3.101) simplifies to,

$$\begin{aligned} \int \mathcal{D}q [(\zeta'_1 + 1)BF_1 + (\zeta'_2 - 1)BF_2 + (\zeta'_3 + 1)\bar{c} sF_1 + (\zeta'_4 - 1)\bar{c} sF_2 \\ + (\zeta_1 - \zeta_3)B(sF_1)\Theta' + (\zeta_2 - \zeta_4)B(sF_2)\Theta'] e^{i(S_{eff} + S_1)} = 0. \end{aligned} \quad (3.109)$$

The comparison of coefficients from the terms of the above equation gives the following constraints on the parameters ζ_i

$$\begin{aligned} \zeta'_1 + 1 = 0, \quad \zeta'_2 - 1 = 0, \quad \zeta'_3 + 1 = 0, \quad \zeta'_4 - 1 = 0, \\ \zeta_1 - \zeta_3 = 0, \quad \zeta_2 - \zeta_4 = 0. \end{aligned} \quad (3.110)$$

The solutions of the above equations satisfying the boundary conditions (3.107) are

$$\zeta_1 = -\kappa, \quad \zeta_2 = \kappa, \quad \zeta_3 = -\kappa, \quad \zeta_4 = \kappa. \quad (3.111)$$

With these values of ζ_i the expression of $S_1[\kappa]$ given in equation (3.106) becomes

$$S_1[\kappa] = \int d\tau [-\kappa B F_1 + \kappa B F_2 - \kappa \bar{c} s F_1 + \kappa \bar{c} s F_2], \quad (3.112)$$

which vanishes at $\kappa = 0$. Now, by adding $S_1(\kappa = 1)$ to the effective action (S_{eff}) given in equation (3.88) we get

$$S_{eff} + S_1(\kappa = 1) = S + \int d\tau \left[\frac{1}{2} B^2 + B F_2[f(q)] + \bar{c} s F_2[f(q)] \right], \quad (3.113)$$

which is nothing but the effective action for relativistic particle models satisfying the different gauge condition $F_2[f(q)] = 0$. Thus, under FCBRST transformation, the generating functional of higher derivative models changes from one gauge condition ($F_1[f(q)] = 0$) to another gauge ($F_2[f(q)] = 0$) as

$$\int d\tau e^{iS_{eff}} \xrightarrow{FCBRST} \left(\int d\tau e^{i[S_{eff} + S_1(\kappa=1)]} \right). \quad (3.114)$$

We end this section by noting that the FCBRST transformation with appropriate finite coordinate-dependent parameter is able to connect two different (arbitrary) gauges of the relativistic particle model.

3.7 Discussions

In this chapter we have taken the relativistic particle model with curvature. Both the massive and massless versions were considered in the first order formalism to explore their gauge symmetries. We have used an extended version of the algorithm of Banerjee et. al. [32] to construct the gauge generators. The apparent mismatch in counting the gauge degrees of freedom from the no. of PFCs [27] has been explained. Also, from the methodology we have shown why the massless version has two gauge degrees of freedom. Moreover, we have extracted both these symmetries using the gauge generator. One of the symmetries was identified with the usual diffeomorphism invariance whereas the other has been shown to be W_3 symmetry. In this connection, it may be noted that this result was earlier obtained only at equations of motion level. The BRST symmetries were explored for both the massive and massless models. The massless version in this case exhibited the W_3 algebra also for the (anti-)BRST.

Chapter 4

Extended Maxwell-Chern-Simons model

In the previous chapter we have demonstrated the application of our method of canonical analysis for particle models. In this chapter, we extend the application of our method to field theories. We consider the action of the extended-Maxwell-Chern-Simon's (EMCS) in 2+1 dimension which is given by

$$S = \int d^3x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{g}{2} \epsilon^{\alpha\beta\gamma} (\partial^\rho \partial_\rho A_\alpha) (\partial_\beta A_\gamma) \right) \quad (4.1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Note that the model is endowed with the $U(1)$ symmetry

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda. \quad (4.2)$$

and is an extension of the usual Maxwell Chern Simons theory [55]. The second term in the action contains higher derivative terms and may be viewed as the higher derivative version of the Chern-Simons piece. The model actually derive its physical significance from the fact that it give rise to the topological gravity in the higher dimensions. The fields in this case are the vector potential A_μ and it's higher time derivatives. The complete Hamiltonian analysis is done in the first order formalism. It is shown that there is only one primary first class constraint along with two secondary first class constraints. Consequently, the gauge generator has three gauge parameters. The exact mapping of this independent gauge symmetry with the $U(1)$ symmetry has been established.

4.1 Hamiltonian analysis of the model in the equivalent first order formalism

In our approach the time derivative of the field A_μ will be considered as additional fields. Thus, it will be convenient to expand the Lagrangian of the model (4.1) in space and time parts. Using the mostly positive metric ($\eta_{\mu\nu} = -, +, +$) the Lagrangian is written as

$$\begin{aligned} \mathcal{L} = &= \frac{1}{2}(\dot{A}_i^2 + (\partial_i A_0)^2 - (\partial_i A_j)^2 - 2\dot{A}_i \partial_i A_0 + \partial_i A_j \partial_j A_i) + \frac{g}{2}\epsilon_{ij}(-\ddot{A}_0 \\ &+ \nabla^2 A_0) \partial_i A_j - \frac{g}{2}\epsilon_{ij}(-\ddot{A}_i + \nabla^2 A_i) \dot{A}_j + \\ &\frac{g}{2}\epsilon_{ij}(-\ddot{A}_i + \nabla^2 A_i) \partial_j A_0 \end{aligned} \quad (4.3)$$

Here the fields are referred to their covariant components and dot represents derivative with respect to time. Note that the effect of the relativistic metric $\eta_{\mu\nu}$ has been taken care of explicitly in writing (4.3). In the following subscripts from the middle of the Greek alphabet μ, ν assume the values 0,1, 2 and those from the middle of the Latin alphabet i, j take values 1 and 2. In any case, they just label the components and no further reference to the relativistic metric is implied.

To analyze the model in the equivalent first order formalism we define the new coordinates

$$\xi_{1\mu} = A_\mu \quad \text{and} \quad \xi_{2\mu} = \dot{A}_\mu \quad (4.4)$$

This immediately imposes the constraint

$$\xi_{2\mu} = \dot{\xi}_{1\mu} \quad (4.5)$$

The equivalent first order Lagrangian is obtained from (4.3) using the definitions (4.4) as

$$\begin{aligned} \mathcal{L}' = & \frac{1}{2}(\xi_{2i}\xi_{2i} + \partial_i \xi_{10} \partial_i \xi_{10} - \partial_i \xi_{1j} \partial_i \xi_{1j} - 2\xi_{2j} \partial_j \xi_{10} + \partial_i \xi_{1j} \partial_j \xi_{1i}) \\ & + \frac{g}{2}\epsilon_{ij}(-\dot{\xi}_{20} + \nabla^2 \xi_{10}) \partial_i \xi_{1j} - \frac{g}{2}\epsilon_{ij}(-\dot{\xi}_{2i} + \nabla^2 \xi_{1i}) \xi_{2j} \\ & + \frac{g}{2}\epsilon_{ij}(-\dot{\xi}_{2i} + \nabla^2 \xi_{1i}) \partial_j \xi_{10} + \xi_{0\mu}(\xi_{2\mu} - \dot{\xi}_{1\mu}) \end{aligned} \quad (4.6)$$

where the constraint (4.5) is enforced by the Lagrange multiplier $\xi_{0\mu}$. Henceforth, $\xi_{0\mu}$ will be considered as independent fields.

To proceed with the canonical analysis we define the momenta $\Pi_{0\mu}, \Pi_{1\mu}, \Pi_{2\mu}$ conjugate to the fields $\xi_{0\mu}, \xi_{1\mu}, \xi_{2\mu}$ respectively in the usual way :

$$\Pi_{\alpha\mu} = \frac{\partial \mathcal{L}'}{\partial \dot{\xi}_{\alpha\mu}} \quad ; \quad \alpha = 0, 1, 2 \quad (4.7)$$

As a result the following primary constraints emerge:

$$\begin{aligned} \Phi_{0\mu} &= \Pi_{0\mu} \approx 0 \\ \Phi_{1\mu} &= \Pi_{1\mu} + \xi_{0\mu} \approx 0 \\ \Phi_{20} &= \Pi_{20} + \frac{g}{2} \epsilon_{ij} \partial_i \xi_{1j} \approx 0 \\ \Phi_{2i} &= \Pi_{2i} - \frac{g}{2} \epsilon_{ij} \xi_{2j} + \frac{g}{2} \epsilon_{ij} \partial_j \xi_{10} \approx 0 \end{aligned} \quad (4.8)$$

The basic Poisson brackets are

$$\{\xi_{\alpha\mu}(\mathbf{x}), \Pi_{\beta\nu}(\mathbf{x}')\} = \delta_{\alpha\beta} \delta_{\mu\nu} \delta^2(\mathbf{x} - \mathbf{x}') \quad (4.9)$$

where $\alpha, \beta = 0, 1, 2$. This leads to the following algebra of the primary constraint,

$$\begin{aligned} \{\Phi_{10}(\mathbf{x}), \Phi_{2i}(\mathbf{x}')\} &= -\frac{g}{2} \epsilon_{ij} \partial_j' \delta^2(\mathbf{x} - \mathbf{x}') \\ \{\Phi_{1i}(\mathbf{x}), \Phi_{20}(\mathbf{x}')\} &= \frac{g}{2} \epsilon_{ij} \partial_j' \delta^2(\mathbf{x} - \mathbf{x}') \\ \{\Phi_{2i}(\mathbf{x}), \Phi_{2j}(\mathbf{x}')\} &= -g \epsilon_{ij} \delta^2(\mathbf{x} - \mathbf{x}') \\ \{\Phi_{0\mu}(\mathbf{x}), \Phi_{1\nu}(\mathbf{x}')\} &= -\delta_{\mu\nu} \delta^2(\mathbf{x} - \mathbf{x}') \end{aligned} \quad (4.10)$$

All other brackets between the constraints vanish. Apparently all the primary constraints have non trivial brackets among themselves. However, we can make the following linear combinations of the primary constraints

$$\begin{aligned} \Phi'_{20} &= \Phi_{20} + \frac{g}{2} \epsilon_{ij} \partial_i \Phi_{0j} \approx 0 \\ \Phi'_{2i} &= \Phi_{2i} + \frac{g}{2} \epsilon_{ij} \partial_j \Phi_{00} \approx 0 \end{aligned} \quad (4.11)$$

Using the algebra of the primary constraints (4.10) we find that the constraint algebra simplifies to

$$\begin{aligned} \{\Phi_{0\mu}(\mathbf{x}), \Phi_{1\nu}(\mathbf{x}')\} &= -\delta_{\mu\nu} \delta^2(\mathbf{x} - \mathbf{x}') \\ \{\Phi'_{2i}(\mathbf{x}), \Phi'_{2j}(\mathbf{x}')\} &= -g \epsilon_{ij} \delta^2(\mathbf{x} - \mathbf{x}') \end{aligned} \quad (4.12)$$

It will thus be convenient to replace the original set of primary constraints $\{\Phi_{0\mu}, \Phi_{10}, \Phi_{1i}, \Phi_{20}, \Phi_{2i}\}$ by $\{\Phi_{0\mu}, \Phi_{10}, \Phi_{1i}, \Phi'_{20}, \Phi'_{2i}\}$. Explicitly, the new set of primary constraints are

$$\begin{aligned}\Phi_{0\mu} &= \Pi_{0\mu} \approx 0 \\ \Phi_{1\mu} &= \Pi_{1\mu} + \xi_{0\mu} \approx 0 \\ \Phi'_{20} &= \Pi_{20} + \frac{g}{2}\epsilon_{ij}\partial_i\xi_{1j} + \frac{g}{2}\epsilon_{ij}\partial_i\Pi_{0j} \approx 0 \\ \Phi'_{2i} &= \Pi_{2i} - \frac{g}{2}\epsilon_{ij}\xi_{2j} + \frac{g}{2}\epsilon_{ij}\partial_j\xi_{10} + \frac{g}{2}\epsilon_{ij}\partial_j\Pi_{00} \approx 0\end{aligned}\quad (4.13)$$

The canonical Hamiltonian is obtained by Legendre transformation as

$$H_{can} = \int \mathcal{H}_{can} d^2\mathbf{x} \quad (4.14)$$

where \mathcal{H}_{can} is the canonical Hamiltonian density, given by:

$$\begin{aligned}\mathcal{H}_{can} &= -\frac{1}{2}(\xi_{2i}\xi_{2i} + \partial_i\xi_{10}\partial_i\xi_{10} - \partial_i\xi_{1j}\partial_i\xi_{1j} - 2\xi_{2j}\partial_j\xi_{10} + \partial_i\xi_{1j}\partial_j\xi_{1i}) - \\ &\quad \frac{g}{2}\epsilon_{ij}\nabla^2\xi_{10}\partial_i\xi_{1j} + \frac{g}{2}\epsilon_{ij}\nabla^2\xi_{1i}\xi_{2j} - \frac{g}{2}\epsilon_{ij}\nabla^2\xi_{1i}\partial_j\xi_{10} - \xi_{0\mu}\xi_{2\mu}\end{aligned}\quad (4.15)$$

The total Hamiltonian is

$$H_T = \int d^2\mathbf{x}(\mathcal{H}_{can} + \Lambda_{0\mu}\Phi_{0\mu} + \Lambda_{1\mu}\Phi_{1\mu} + \Lambda_{20}\Phi'_{20} + \Lambda_{2i}\Phi'_{2i}) \quad (4.16)$$

The multipliers $\Lambda_{0\mu}$, $\Lambda_{1\mu}$, and $\Lambda_{2\mu}$ are arbitrary at this stage.

The primary constraints (4.13) should be conserved in time, i.e. there Poisson bracket with H_T should vanish. Conserving $\Phi_{0\mu}$, $\Phi_{1\mu}$, Φ'_{2i} in time the following multipliers are fixed,

$$\begin{aligned}\Lambda_{00} &= \nabla^2\xi_{10} - \partial_i\xi_{2i} - g\epsilon_{ij}\nabla^2\partial_i\xi_{1j} \\ \Lambda_{0i} &= -\nabla^2\xi_{1i} + \partial_i\partial_j\xi_{1j} - g\epsilon_{ij}\nabla^2\partial_j\xi_{10} + \frac{g}{2}\epsilon_{ij}\xi_{2j} \\ \Lambda_{1\mu} &= \xi_{2\mu} \\ \Lambda_{2i} &= \frac{1}{2}(\nabla^2\xi_{1i} + \partial_i\xi_{20}) + \frac{1}{g}\epsilon_{ij}(\partial_j\xi_{10} - \xi_{0j} - \xi_{2j})\end{aligned}\quad (4.17)$$

Only Λ_{20} remains arbitrary. Substituting these in the total Hamiltonian we find that it contains only one arbitrary multiplier Λ_{20} . This shows that

there is only one gauge degree of freedom, a result consistent with the gauge transformation of the original field A_μ .

Conserving Φ'_{20} in time, a secondary constraint emerges.

$$\Psi_1 = \xi_{00} + \frac{g}{2}\epsilon_{ij}\partial_i\xi_{2j} \approx 0 \quad (4.18)$$

From $\dot{\Psi}_1 = 0$ we get

$$\{\Psi_1, H_T\} = 0 \quad (4.19)$$

A straightforward calculation gives

$$\Lambda_{00} - g\epsilon_{ij}\partial_j\Lambda_{2i} = 0 \quad (4.20)$$

Using the values of Λ_{00} and Λ_{2i} from (4.17) and simplifying we get

$$\partial_i\xi_{0i} - \frac{g}{2}\epsilon_{ij}\nabla^2\partial_i\xi_{1j} = 0 \quad (4.21)$$

which is a new secondary constraint

$$\Psi_2 = \partial_i\xi_{0i} - \frac{g}{2}\epsilon_{ij}\nabla^2\partial_i\xi_{1j} \approx 0 \quad (4.22)$$

The condition $\{\Psi_2, H_T\} = 0$ gives

$$\partial_i\Lambda_{0i} - \frac{g}{2}\epsilon_{ij}\nabla^2\partial_i\Lambda_{1j} = 0 \quad (4.23)$$

Substituting the values of Λ_{0i} and Λ_{1j} the above equation reduces to the form $0 = 0$. Hence the iterative process stops here giving no further constraints. The primary constraints of the theory are $\{\Phi_{0\mu}, \Phi_{1\mu}, \Phi'_{20}, \Phi'_{2i}\}$ while the secondary constraints are Ψ_1 and Ψ_2 .

Using the Poisson brackets (4.9) the complete algebra of constraints can be worked out as

$$\begin{aligned} \{\Phi_{0\mu}(\mathbf{x}), \Phi_{1\nu}(\mathbf{x}')\} &= -\delta_{\mu\nu}\delta^2(\mathbf{x} - \mathbf{x}') \\ \{\Phi'_{2i}(\mathbf{x}), \Phi'_{2j}(\mathbf{x}')\} &= -g\epsilon_{ij}\delta^2(\mathbf{x} - \mathbf{x}') \\ \{\Psi_1(\mathbf{x}), \Phi_{0\nu}(\mathbf{x}')\} &= \delta_{0\nu}\delta^2(\mathbf{x} - \mathbf{x}') \\ \{\Psi_1(\mathbf{x}), \Phi'_{2i}(\mathbf{x}')\} &= -g\epsilon_{ij}\partial_j\delta^2(\mathbf{x} - \mathbf{x}') \\ \{\Psi_2(\mathbf{x}), \Phi_{0\mu}(\mathbf{x}')\} &= \delta_{\mu i}\partial_i\delta^2(\mathbf{x} - \mathbf{x}') \\ \{\Psi_2(\mathbf{x}), \Phi_{1\mu}(\mathbf{x}')\} &= -\frac{g}{2}\epsilon_{ij}\delta_{j\mu}\nabla^2\partial_i\delta^2(\mathbf{x} - \mathbf{x}') \end{aligned} \quad (4.24)$$

The constraint algebra appears to be complicated but new linear combinations will simplify the algebra. Before going into that discussion it is time to get rid of the unphysical variables $\xi_{0\mu}$ and $\Pi_{0\mu}$.

4.1.1 Calculation in reduced phase space

The fields $\xi_{0\mu}$ and $\Pi_{0\mu}$ can be eliminated by strongly imposing the constraints $\Phi_{0\mu}$ and $\Phi_{1\mu}$ ¹. The remaining constraints of the theory can now be rewritten as

$$\begin{aligned}
\Phi_{20} &= \Pi_{20} + \frac{g}{2}\epsilon_{ij}\partial_i\xi_{1j} \approx 0 \\
\Phi_{2i} &= \Pi_{2i} - \frac{g}{2}\epsilon_{ij}(\xi_{2j} - \partial_j\xi_{10}) \approx 0 \\
\Psi_1 &= -\Pi_{10} + \frac{g}{2}\epsilon_{ij}\partial_i\xi_{2j} \approx 0 \\
\Psi_2 &= -\partial_i\Pi_{1i} - \frac{g}{2}\epsilon_{ij}\nabla^2\partial_i\xi_{1j} \approx 0
\end{aligned} \tag{4.25}$$

The Poisson brackets between these constraints can be read from (4.24). The nontrivial brackets are

$$\begin{aligned}
\{\Phi_{2i}(\mathbf{x}), \Phi_{2j}(\mathbf{x}')\} &= -g\epsilon_{ij}\delta^2(\mathbf{x} - \mathbf{x}') \\
\{\Psi_1(\mathbf{x}), \Phi_{2i}(\mathbf{x}')\} &= -g\epsilon_{ij}\partial_j\delta^2(\mathbf{x} - \mathbf{x}')
\end{aligned} \tag{4.26}$$

We can form the linear combination

$$\Psi'_1 = \Psi_1 + \partial_i\Phi_{2i} \tag{4.27}$$

It can be easily checked that Ψ'_1 has vanishing brackets with all other constraints. Replacing the set of constraints $\{\Phi_{20}, \Phi_{2i}, \Psi_1, \text{ and } \Psi_2\}$ by the new set $\{\Phi_{20}, \Phi_{2i}, \Psi'_1, \text{ and } \Psi_2\}$, we find that there are three first class constraints: $\Phi_{20}, \Psi'_1, \text{ and } \Psi_2$ and two second-class constraints Φ_{2i} . The classification of the constraints of the theory is tabulated in Table 1.

Before proceeding further, a degrees of freedom count will be instructive. The total number of phase space variables is 12. There are three first class constraints and two second-class constraints. Hence, the no of degrees of freedom is

$$12 - (2 \times 3 + 2) = 4$$

We find that the number of degrees of freedom is doubled compared with the Maxwell theory, which is expected due to the higher derivative nature [26].

¹Technically this should be done by replacing the Poisson brackets by the corresponding Dirac brackets. However, the Dirac brackets here are trivial i.e. the Dirac brackets between the remaining phase space variables are the same as the Poisson brackets.

Table 4.1: Classification of Constraints of the model (4.1)

	First class	Second class
Primary	Φ_{20}	Φ_{2i}
Secondary	Ψ'_1, Ψ_2	

4.1.2 Reduction of second class constraints

After the elimination of the unphysical sector ($\xi_{0\mu}, \Pi_{0\mu}$), the total Hamiltonian becomes

$$H_T(\mathbf{x}) = \int d^2\mathbf{x} (\mathcal{H}_{can}(\mathbf{x}) + \Lambda_{20}(\mathbf{x})\Phi_{20}(\mathbf{x}) + \Lambda_{2i}(\mathbf{x})\Phi_{2i}(\mathbf{x})) \quad (4.28)$$

where \mathcal{H}_{can} is the canonical Hamiltonian density given by

$$\begin{aligned} \mathcal{H}_{can} = & -\frac{1}{2}(\xi_{2i}\xi_{2i} + \partial_i\xi_{10}\partial_i\xi_{10} - \partial_i\xi_{1j}\partial_i\xi_{1j} - 2\xi_{2j}\partial_j\xi_{10} + \partial_i\xi_{1j}\partial_j\xi_{1i}) \\ & -\frac{g}{2}\epsilon_{ij}\nabla^2\xi_{10}\partial_i\xi_{1j} + \frac{g}{2}\epsilon_{ij}\nabla^2\xi_{1i}(\xi_{2j} - \partial_j\xi_{10}) + \Pi_{1\mu}\xi_{2\mu} \end{aligned} \quad (4.29)$$

and

$$\Lambda_{2i} = \frac{1}{2}(\nabla^2\xi_{1i} + \partial_i\xi_{20}) + \frac{1}{g}\epsilon_{ij}(\partial_j\xi_{10} + \Pi_{1j} - \xi_{2j})$$

Λ_{20} is arbitrary. It signifies that there is one continuous gauge degree of freedom.

In the next section we will explicitly construct the gauge generator using the method given in [29]. Since the method is directly applicable to theories with first class constraint only, we have to eliminate the second class constraints of our theory. Following Dirac's method of constraint Hamiltonian analysis we can strongly put the second class constraints to be zero if the Poisson brackets are replaced by the corresponding Dirac brackets.

The field theoretic version of the Dirac brackets 2.15 between two phase space variables A and B is can be written as

$$\begin{aligned} \{A(\mathbf{x}), B(\mathbf{x}')\}_D = & \{A(\mathbf{x}), B(\mathbf{x}')\} - \int \{A(\mathbf{x}), \Phi_{2i}(\mathbf{y})\} \\ & \Delta_{ij}^{-1}(\mathbf{y}, \mathbf{z}) \{\Phi_{2j}(\mathbf{z}), B(\mathbf{x}')\} d^2\mathbf{y}d^2\mathbf{z} \end{aligned} \quad (4.30)$$

where $\Delta_{ij}^{-1}(\mathbf{x}, \mathbf{x}')$ is the inverse of the matrix

$$\Delta_{ij}(\mathbf{x}, \mathbf{x}') = \{\Phi_{2i}(\mathbf{x}), \Phi_{2j}(\mathbf{x}')\} \quad (4.31)$$

The nontrivial Dirac brackets between the phase space variables are calculated as

$$\begin{aligned} \{\xi_{1\mu}(\mathbf{x}), \Pi_{1\nu}(\mathbf{x}')\}_D &= \delta_{\mu\nu} \delta^2(\mathbf{x} - \mathbf{x}') \\ \{\xi_{2i}(\mathbf{x}), \xi_{2j}(\mathbf{x}')\}_D &= \frac{1}{g} \epsilon_{ij} \delta^2(\mathbf{x} - \mathbf{x}') \\ \{\xi_{2i}(\mathbf{x}), \Pi_{2j}(\mathbf{x}')\}_D &= \frac{1}{2} \delta_{ij} \delta^2(\mathbf{x} - \mathbf{x}') \\ \{\xi_{2i}(\mathbf{x}), \Pi_{10}(\mathbf{x}')\}_D &= -\frac{1}{2} \partial'_i \delta^2(\mathbf{x} - \mathbf{x}') \\ \{\Pi_{2i}(\mathbf{x}), \Pi_{10}(\mathbf{x}')\}_D &= \frac{g}{4} \epsilon_{ij} \partial'_j \delta^2(\mathbf{x} - \mathbf{x}') \\ \{\Pi_{2i}(\mathbf{x}), \Pi_{2j}(\mathbf{x}')\}_D &= \frac{g}{4} \epsilon_{ij} \delta^2(\mathbf{x} - \mathbf{x}') \end{aligned} \quad (4.32)$$

All other Dirac brackets are the same as the corresponding Poisson brackets.

4.2 Gauge symmetry in EMCS

As has been mentioned earlier we will follow the method of [29] to construct the gauge generator containing the exact number of independent gauge parameters. The essence of the method has been reviewed in Sec. 2. Accordingly, we rename the constraints as $\Omega_1 = \Phi_{20}$, $\Omega_2 = \Psi'_1$ and $\Omega_3 = \Psi_2$. The gauge generator is

$$G = \int \epsilon_a \Omega_a d^2 \mathbf{x} \quad (4.33)$$

which is a field theoretic extension of (2.16). These structure functions are now defined by

$$\begin{aligned} \{H_{can}, \Omega_a(\mathbf{x})\}_D &= \int d^2 \mathbf{y} V_{ab}(\mathbf{y}, \mathbf{x}) \Omega_b(\mathbf{y}) \\ \{\Omega_a(\mathbf{x}), \Omega_b(\mathbf{y})\}_D &= \int d^2 \mathbf{z} C_{abc}(\mathbf{z}, \mathbf{x}, \mathbf{y}) \Omega_c(\mathbf{z}) \end{aligned} \quad (4.34)$$

and the master equation (2.18) takes the form

$$\begin{aligned} 0 &= \frac{d\epsilon_{a_1}(\mathbf{x})}{dt} - \int d^2 \mathbf{y} \epsilon_b(\mathbf{y}) V_{ba_1}(\mathbf{x}, \mathbf{y}) - \\ &\quad \int d^2 \mathbf{y} d^2 \mathbf{z} \epsilon_b(\mathbf{y}) \Lambda_{c_1}(\mathbf{z}) C_{c_1 ba_1}(\mathbf{z}, \mathbf{y}, \mathbf{x}) \end{aligned} \quad (4.35)$$

Note that Dirac brackets appear on the left hand sides of Eq. (4.34). This is because there were second class constraints in our theory which have been eliminated by the Dirac bracket formalism.

Using the defining relations (4.34) and the Dirac brackets (4.32) we find that the only nonvanishing V_{ab} are given by

$$\begin{aligned} V_{12}(\mathbf{x}, \mathbf{y}) &= -\delta^2(\mathbf{x} - \mathbf{y}) \\ V_{23}(\mathbf{x}, \mathbf{y}) &= -\delta^2(\mathbf{x} - \mathbf{y}) \end{aligned} \quad (4.36)$$

Similarly, from the algebra of the constraints we find all $C_{abc} = 0$. Substituting these values in the Eq.(4.35) we get the following conditions on the gauge parameters ϵ_a :

$$\begin{aligned} \dot{\epsilon}_2 + \epsilon_1 &= 0 \\ \dot{\epsilon}_3 + \epsilon_2 &= 0 \end{aligned} \quad (4.37)$$

Solving these, we find

$$\begin{aligned} \epsilon_1 &= \ddot{\epsilon}_3 \\ \epsilon_2 &= -\dot{\epsilon}_3 \end{aligned} \quad (4.38)$$

Hence the desired gauge generator assumes the form

$$G = \int d^2x (\ddot{\epsilon}_3 \Omega_1 - \dot{\epsilon}_3 \Omega_2 + \epsilon_3 \Omega_3) \quad (4.39)$$

It is immediately observed that G contains one arbitrary gauge parameter, namely ϵ_3 .

We still have the additional restrictions (2.39). In our case this leads to the condition

$$\delta\xi_{2\mu} = \frac{d}{dt}\delta\xi_{1\mu} \quad (4.40)$$

where $\delta\xi_{1\mu}$, $\delta\xi_{2\mu}$ are the gauge variations of $\xi_{1\mu}$ and $\xi_{2\mu}$ respectively. Using the generator G (4.39) we get

$$\delta\xi_{2\mu} = \{\xi_{2\mu}, G\}_D = \partial_\mu \dot{\epsilon}_3 \quad (4.41)$$

Similarly

$$\delta\xi_{1\mu} = \{\xi_{1\mu}, G\}_D = \partial_\mu \epsilon_3 \quad (4.42)$$

Clearly the additional restriction (4.40) is identically satisfied. Thus, no more restriction is imposed on the gauge parameters.

Finally we look at the comparison of the transformations generated by the Hamiltonian gauge generator with Lagrangian gauge symmetry (4.2). Since $\xi_{1\mu} = A_\mu$ we have

$$\delta A_\mu = \partial_\mu \epsilon_3 \quad (4.43)$$

from (4.42). This is the same transformation as (4.2) if we put $\epsilon_3 = \lambda$.

4.3 Discussions

In this chapter, we have considered the extended Maxwell Chern Simons (EMCS) model as an example to illustrate the application of the algorithm developed in this thesis to a higher derivative field theoretic model. The EMCS model was introduced by Deser and is an extension of the usual Maxwell Chern Simons model [55]. The first order formalism which we are using singles out the time derivative of the fields as additional ‘coordinate’. Due to this, a manifest Lorentz covariance is lost. We have therefore, converted the entire theory in terms of covariant components and the effect of the relativistic metric was taken care of by explicitly writing the effective first order Lagrangian.

A detailed constraint analysis from the first order Lagrangian has been performed. We obtained three first class constraints and a pair of second class constraint. An usual degree of freedom count demonstrated that there was a doubling of freedom, compared with the usual MCS theory. This result is consistent with the higher derivative nature of the model.

As has been discussed earlier, the algorithm for constructing independent gauge generator of the model developed in this thesis is applicable for theories with first class constraints only. We have therefore reduced the phase space by replacing the Poisson brackets by the corresponding Dirac brackets. This enables one to put the second class constraints strongly equal to zero. The model has been treated in the first order formalism. Since it is a higher derivative theory the degrees of freedom count shows that the phase space has been doubled. Finally, we show that the gauge generator obtained by us generate $U(1)$ gauge transformation of the phase space variables.

Chapter 5

The quantum charged rigid membrane

We are continuing the application of our method to different types of systems. In this chapter we will consider the example of charged rigid membrane. The action for this model will be constructed from the extrinsic curvature of the membrane evolving in a Minkowski space time.

Theories with extrinsic curvatures are frequently studied, especially in string theory. Although this concept is not new, but recent inclusion of these theories in some physically interesting models added an extra urgency to revisit their symmetry features. As proposed by Dirac in 1962, the membrane model of the electron is a charged bubble, evolving with time. The model is not a physical one but has important consequences on brane related theories. To have a basic knowledge of the membrane we have taken, consider the evolving surface Σ in a background Minkowski spacetime $\eta^{\mu\nu}$ ¹. The surface is described by the local coordinate x^μ of the background spacetime. The embedding function $X^\mu(\xi^a) = x^\mu$ is a function of the local coordinates of the world volume m , swept out by the surface. We consider the following effective action underlying the dynamics of the surface Σ [77]:

$$S[X^\mu] = \int_m d^3\xi (-\alpha K + \beta j^a e^\mu_a A_\mu), \quad (5.1)$$

where $K = g^{ab} K_{ab}$ being the extrinsic curvature² and α, β are constant related to the rigidity parameter and form factor respectively. On the other hand, j^a which minimally couples the charged surface and the electromagnetic field A_μ [78], is a constant electric current density distributed over the

¹with $\mu, \nu = 0, 1, 2, 3$ and $a, b = 0, 1, 2$

² g_{ab} is the worldvolume metric and $e^\mu_a = X^\mu_{,a}$ are tangent vectors to the worldvolume

world volume and is locally conserved on m with $\partial_a j^a = 0$. Variation of the action with respect to the embedding function $X^\mu(\xi^a)$ leads to the equation of motion

$$\alpha \mathcal{R} = \frac{\beta}{\sqrt{-g}} j^a n^\mu e^\nu{}_a F_{\mu\nu}. \quad (5.2)$$

The above equation (5.2) can be thought as a Lorentz force equation with \mathcal{R} being the Gaussian curvature and $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$ the electromagnetic field tensor. Under suitable choice of the embedding functions $(X^\mu(\tau, \theta, \varphi) = (t(\tau), r(\tau), \theta, \varphi))$ equation (5.1) boils down to [77]

$$S = 4\pi \int d\tau L(r, \dot{r}, \ddot{r}, \dot{t}, \ddot{t}) \quad (5.3)$$

where the Lagrangian L , which is higher derivative in nature is given by,

$$L = -\frac{\alpha r^2}{\dot{t}^2 - \dot{r}^2} (\ddot{r}\dot{t} - \dot{r}\ddot{t}) - 2\alpha r\dot{t} - \frac{\beta q^2 \dot{t}}{r}. \quad (5.4)$$

Promptly, we can write down the equation of motion for the higher derivative Lagrangian:

$$\frac{d}{d\tau} \left(\frac{\dot{r}}{\dot{t}} \right) = -\frac{\dot{t}^2 - \dot{r}^2}{2r\dot{t}^3} \left(\dot{t}^2 - \frac{\beta(\dot{t}^2 - \dot{r}^2)^2 q^2}{2\alpha r^2} \right). \quad (5.5)$$

Lagrangian (5.4) will be our sole interest. The model is reparametrisation invariant under the parameter $\tau \rightarrow \tau + \sigma$, with σ being infinitesimal. In the proceeding sections we will develop the Hamiltonian formulation for this model.

5.1 Hamiltonian analysis

Before we start the Hamiltonian analysis we need to convert the higher derivative Lagrangian (5.4) to a first order lagrangian, named as the auxiliary lagrangian, by introduction of the new fields

$$\begin{aligned} \dot{r} &= R \\ \dot{t} &= T \end{aligned} \quad (5.6)$$

So, we write down the auxiliary Lagrangian as ³

$$L' = -\frac{\alpha r^2}{N^2} (\dot{R}T - R\dot{T}) - 2\alpha rT - \frac{\beta q^2 T}{r} + \lambda_1(R - \dot{r}) + \lambda_2(T - \dot{t}) \quad (5.7)$$

³consider $N^2 = T^2 - R^2$, for convenience

Inclusion of new fields impose constraints

$$R - \dot{r} \approx 0, \quad T - \dot{t} \approx 0 \quad (5.8)$$

which are taken care of via the multipliers λ_1 and λ_2 . Variation of L' with respect to r, R, t, T, λ_1 and λ_2 give rise to the following equation of motions:

$$-\frac{2\alpha r}{N^2}(\dot{R}T - R\dot{T}) - 2\alpha T + \frac{\beta q^2 T}{r^2} + \dot{\lambda}_1 = 0 \quad (5.9)$$

$$-\frac{2\alpha r^2}{N^4}R(\dot{R}T - R\dot{T}) + \frac{d}{d\tau} \left(\frac{\alpha r^2}{N^2}T \right) + \frac{\alpha r^2}{N^2}\dot{T} + \lambda_1 = 0 \quad (5.10)$$

$$\dot{\lambda}_2 = 0 \quad (5.11)$$

$$\frac{2\alpha r^2}{N^4}T(\dot{R}T - R\dot{T}) - \frac{d}{d\tau} \left(\frac{\alpha r^2}{N^2}R \right) - \frac{\alpha r^2}{N^2}\dot{R} - 2\alpha r - \frac{\beta q^2}{r} + \lambda_2 = 0 \quad (5.12)$$

$$R - \dot{r} = 0 \quad (5.13)$$

$$T - \dot{t} = 0 \quad (5.14)$$

(5.14), (5.14) are obvious since they correspond to (5.8).

Before proceeding for Hamiltonian formulation, we identify the new phase space which is constituted of the variables are $(r, \Pi_r), (t, \Pi_t), (R, \Pi_R), (T, \Pi_T), (\lambda_1, \Pi_{\lambda_1}), (\lambda_2, \Pi_{\lambda_2})$. Here $\Pi_{x^\mu} = \frac{\partial L'}{\partial \dot{x}^\mu}$, are the momenta corresponding to x^μ which generically stands for the variables $r, R, t, T, \lambda_1, \lambda_2$. We immediately obtain the primary constraints as listed bellow

$$\begin{aligned} \Phi_1 &= \Pi_r + \lambda_1 \approx 0 \\ \Phi_2 &= \Pi_t + \lambda_2 \approx 0 \\ \Phi_3 &= \Pi_R + \frac{\alpha r^2}{N^2}T \approx 0 \\ \Phi_4 &= \Pi_T - \frac{\alpha r^2}{N^2}R \approx 0 \\ \Phi_5 &= \Pi_{\lambda_1} \approx 0 \\ \Phi_6 &= \Pi_{\lambda_2} \approx 0 \end{aligned} \quad (5.15)$$

The poisson brackets between the field variables are defined as:

$$\begin{aligned} \{x^\mu, \Pi_{x^\nu}\} &= \delta_{\mu\nu} \\ \{x^\mu, x^\nu\} &= \{\Pi_{x^\mu}, \Pi_{x^\nu}\} = 0 \end{aligned} \quad (5.16)$$

With the aid of (5.16) the non zero Poisson brackets between the primary

constraints can be written down

$$\begin{aligned}\{\Phi_1, \Phi_3\} &= -\frac{2\alpha r}{N^2}T \\ \{\Phi_1, \Phi_4\} &= \frac{2\alpha r}{N^2}R \\ \{\Phi_1, \Phi_5\} &= 1 \\ \{\Phi_2, \Phi_6\} &= 1\end{aligned}\tag{5.17}$$

We can take the following combination of the constraints

$$\Phi'_3 = R\Phi_3 + T\Phi_4 \approx 0\tag{5.18}$$

$$\Phi'_4 = \Phi_4 - \frac{2\alpha r R}{N^2}\Phi_5 \approx 0\tag{5.19}$$

so that the new set of primary constraints are $\Phi_1, \Phi_2, \Phi'_3, \Phi'_4, \Phi_5, \Phi_6$. The complete algebra of primary constraints is now given by (only the nonzero brackets are listed),

$$\{\Phi_1, \Phi_5\} = \{\Phi_2, \Phi_6\} = 1\tag{5.20}$$

The canonical Hamiltonian via Legendre transformation takes the form as

$$H_{can} = 2\alpha r T + \frac{\beta q^2 T}{r} - \lambda_1 R - \lambda_2 T.\tag{5.21}$$

The total Hamiltonian is

$$H_T = H_{can} + \Lambda_1 \Phi_1 + \Lambda_2 \Phi_2 + \Lambda_3 \Phi'_3 + \Lambda_4 \Phi'_4 + \Lambda_5 \Phi_5 + \Lambda_6 \Phi_6\tag{5.22}$$

Here $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5, \Lambda_6$ are the Lagrange multipliers which are arbitrary at this stage. Only those multipliers which are attached to the primary second-class constraints will be determined, others corresponding to primary first class constraints will remain undetermined (although they can be determined too via equation of motion). At this level, loosely speaking Φ'_3 and Φ'_4 are first class constraints (this classification may be changed after we get the full list of constraints). These two may provide us two new secondary constraints and the list can still keep increasing until we get all the constraints. Now, we move towards extracting all constraints of this system. This can be done by demanding that Poisson brackets of the constraints with the total Hamiltonian (time evolution) of the constraints is zero. Preserving $\Phi_1, \Phi_2, \Phi_5, \Phi_6$ in time solves the following multipliers respectively

$$\begin{aligned}\Lambda_5 &= 2\alpha T - \frac{\beta q^2 T}{r^2} \\ \Lambda_6 &= 0 \\ \Lambda_1 &= R \\ \Lambda_2 &= T.\end{aligned}\tag{5.23}$$

Whereas, time conservation of the primary constraints Φ'_3 and Φ'_4 leads to the secondary constraints Ψ_1 and Ψ_2 respectively given by

$$\begin{aligned}\Psi_1 &= -2\alpha r T - \frac{\beta q^2 T}{r} + \lambda_1 R + \lambda_2 T \approx 0 \\ \Psi_2 &= -2\alpha r - \frac{\beta q^2}{r} + \lambda_2 - \frac{2\alpha r}{N^2} R^2 \approx 0\end{aligned}\quad (5.24)$$

Before proceeding further we list below all the nonzero Poisson brackets of the secondary constraints Ψ_1, Ψ_2 with other constraints:

$$\begin{aligned}\{\Phi_1, \Psi_1\} &= 2\alpha T - \frac{\beta q^2 T}{r^2} \\ \{\Phi_5, \Psi_1\} &= -R \\ \{\Phi_6, \Psi_1\} &= -T \\ \{\Phi_1, \Psi_2\} &= \frac{2\alpha}{N^2} T^2 - \frac{\beta q^2}{r^2} \\ \{\Phi'_4, \Psi_2\} &= -\frac{4\alpha r}{N^4} T R^2 \\ \{\Phi_6, \Psi_2\} &= -1\end{aligned}\quad (5.25)$$

Now, time preservation of the secondary constraint Ψ_1 gives identically $0 = 0$. And requirement of $\dot{\Psi}_2 = 0$ solves the Lagrange multiplier $\Lambda_4 = -\frac{A}{B}R$, with $A = \frac{2\alpha T^2}{N^2} - \frac{\beta q^2}{r^2}$ and $B = -\frac{4\alpha r T R^2}{N^4}$.

From the constraint algebra (5.20) and (5.25) one can clearly assert that there is only one first class constraint Φ'_3 with seven other second class constraints $\Phi_1, \Phi_2, \Phi'_4, \Phi_5, \Phi_6, \Psi_1, \Psi_2$. One point worth noting since there are odd number of second class constraints, it indicate there might be some other first class constraint to make the pair of second class constraints even. Judiciously, we can choose a combination $\Psi'_1 = \Psi_1 - \Lambda_1 \Phi_1 - \Lambda_2 \Phi_2 - \Lambda_4 \Phi'_4 - \Lambda_5 \Phi_5 - \Lambda_6 \Phi_6$ so that the pair (Φ'_3, Ψ'_1) becomes first-class. This completes our constraint classification.

Having completed the constraint classification, its time to get rid of the unphysical sector $(\lambda_1, \Pi_{\lambda_1})$ and $(\lambda_2, \Pi_{\lambda_2})$ by imposing the primary second class constraints $\Phi_1, \Phi_2, \Phi_5, \Phi_6$ strongly zero. This can be done by replacing all Poisson brackets by Dirac brackets for rest of the calculations. Surprisingly, Dirac brackets between the basic fields remain same as their corresponding Poisson brackets. So, now our phase space is spanned by $\{r, \Pi_r, t, \Pi_t, R, \Pi_R, T, \Pi_T\}$. For convenience of future calculations we re-

name the constraints as

$$F_1 = \Phi'_3 = R\Phi_3 + T\Phi_4 \approx 0 \quad (5.26)$$

$$F_2 = \Psi_1 - \Lambda_4\Phi_4 \approx 0 \quad (5.27)$$

$$S_1 = \Phi_4 \approx 0 \quad (5.28)$$

$$S_2 = \Psi_2 = -\Pi_t - 2\alpha r - \frac{\beta q^2}{r} - \frac{2\alpha r R^2}{N^2} \approx 0. \quad (5.29)$$

Here, F_1, F_2 is the first class pair with F_1 as primary first class constraint. So far we observed that in this theory, there is only one primary first class constraint with one undetermined multiplier which clearly indicate existence of gauge symmetry(s) in the system. In the next section we will extract the gauge symmetries of this quantum charged rigid membrane.

5.2 Gauge symmetry and Virasoro algebra

To study gauge symmetry we need to remove all the second class constraint from the system by setting them strongly zero and performing Dirac bracket defined by (2.15). To compute Δ_{ij}^{-1} for the set of second class constraints, we have $\{S_1, S_2\} = -\frac{4\alpha r R^2}{N^4}$. So, we can compute the Dirac Brackets be-

tween the basic fields. The nonzero DBs are ⁴:

$$\begin{aligned}
\{r, \Pi_r\}_D &= 1 \\
\{\Pi_r, t\}_D &= -\frac{N^2}{2TR} \\
\{\Pi_r, T\}_D &= -\frac{A}{B} \\
\{\Pi_r, \Pi_T\}_D &= -\frac{2\alpha r R}{N^2} + \frac{Ar}{2R} \\
\{t, \Pi_t\}_D &= 1 \\
\{t, \Pi_R\}_D &= \frac{r(T^2 + R^2)}{4TR^2} \\
\{t, T\}_D &= -\frac{1}{B} \\
\{t, \Pi_T\}_D &= -\frac{r}{2R} \\
\{R, \Pi_R\}_D &= 1 \\
\{\Pi_R, T\}_D &= -\frac{T}{R} \\
\{\Pi_r, \Pi_R\}_D &= \frac{2\alpha r T}{N^2} + \frac{A}{B} \frac{\alpha r^2 (T^2 + R^2)}{N^4} \\
\{\Pi_R, \Pi_T\}_D &= \frac{\alpha r^2}{N^2}
\end{aligned} \tag{5.30}$$

The generator of the gauge transformation is given by a linear combination of all first class constraints,

$$G = \epsilon_1 F_1 + \epsilon_2 F_2 \tag{5.31}$$

where ϵ_1 and ϵ_2 are gauge parameters. We need to find out whether these gauge parameters are independent or not.

The Dirac brackets between the first class constraints are given by

$$\{F_i, F_j\}_D = -\epsilon_{ij} F_2 \quad ; \quad i, j = 1, 2 \tag{5.32}$$

Using a suggestive notation we rename the constraints F_1 and F_2 as

$$L_0 = F_1 \tag{5.33}$$

$$L_1 = F_2 \tag{5.34}$$

We can easily identify a sort of truncated Virasoro algebra of the form

$$\{L_m, L_n\}_D = (m - n)L_{m+n} \tag{5.35}$$

⁴with $A = \frac{2\alpha T^2}{N^2} - \frac{\beta q^2}{r^2}$, $B = \frac{\alpha r T R^2}{N^4}$

with $m = 0$, $n = 1$ as proposed in [76] for higher derivative cases.

Now, using equations (2.19, 2.20) we compute the structure constants as $C_{122} = -1 = -C_{212}$ and $V_{12} = 1$ (other structure constants are zero). Exploiting the master equations (2.18) we find the the following relation between the gauge parameters

$$\epsilon_1 = -\Lambda_3 \epsilon_2 - \dot{\epsilon}_2. \quad (5.36)$$

It is clear that we have only one independent gauge symmetry in this system which is supported by the fact that there is only one undetermined multiplier. We consider ϵ_2 to be independent and compute the gauge transformation of the fields

$$\delta r = -\epsilon_2 R \quad (5.37)$$

$$\delta t = -\epsilon_2 T \quad (5.38)$$

$$\delta R = \epsilon_1 R \quad (5.39)$$

$$\delta T = \epsilon_1 T + \epsilon_2 \frac{A}{B} R \quad (5.40)$$

We can identify this gauge symmetry as reparametrisation symmetry in the following manner. Consider an infinitesimal transformation of r and t on the worldvolume as $\tau \rightarrow \tau + \sigma$. For some infinitesimal σ , we can write

$$\begin{aligned} \delta r &= -\sigma r \\ \delta t &= -\sigma t \end{aligned} \quad (5.41)$$

Clearly, a comparison between (5.37, 6.40) and both equations of (5.41) shows that the reparametrisation parameter is given by $\sigma = \epsilon_2$. Using (5.41) we compute of Gauge variation of the Lagrangian (5.4) which simplifies to

$$\delta L = \frac{d}{d\tau}(\sigma L) \quad (5.42)$$

and ensure the invariance of the action under (5.41).

5.3 Consistency check

It would be worth to find out the Hamiltonian equations of motion which are given by

$$\dot{r} = R \quad (5.43)$$

$$\dot{t} = T \quad (5.44)$$

$$\dot{R} = \Lambda_3 R \quad (5.45)$$

$$\dot{T} = -\frac{A}{B} R + \frac{\dot{R}}{R} T \quad (5.46)$$

Equations (5.43) and (5.44) are obvious as they arise as constraints at the Lagrangian level and agrees with (5.14) and (5.14). Taking time derivative of (5.37) and (6.40) we get

$$\frac{d}{d\tau}\delta r = -\dot{\epsilon}_2 R - \epsilon_2 \dot{R} \quad (5.47)$$

$$\frac{d}{d\tau}\delta t = -\dot{\epsilon}_2 T - \epsilon_2 \dot{T} \quad (5.48)$$

Using equation (6.38) alongwith (5.45, 5.46) the above equations (5.47, 5.48) simplify to

$$\frac{d}{d\tau}\delta r = \delta R \quad (5.49)$$

$$\frac{d}{d\tau}\delta t = \delta T \quad (5.50)$$

which is a direct verification for (2.39). Whereas, (5.46) along with the trivial equation of motions (5.43) and (5.44) can be cast into the form so that it verify (5.5). This indeed is an important outcome of this analysis which agrees the validity of this first order formalism via matching the equation of motion at higher derivative and first order level.

Taking gauge variation of the equation of (5.45) and using (5.39)we get

$$\delta\Lambda_3 = \dot{\epsilon}_1 \quad (5.51)$$

which in turn verifies the first master equation (2.17).

5.4 Discussions

In this chapter, we considered the quantum charged rigid membrane, which actually was proposed by Dirac [58]. The Lagrangian for this model contains higher derivatives. The model is important because it can serve as a toy model for brane related theories like the geodetic brane gravity. We have taken this model to apply the first order formalism as it belongs to a very important class of physics. For Hamiltonian formulation, the Lagrangian is converted into the first order form and we obtained the Lagrangian equations of motion with respect to the dynamical variables. We followed the programme developed in [29] to unveil the gauge symmetries of this model. We found that there is only one independent primary first class constraint which is consistent with the fact that there is only one independent gauge degree of freedom. The gauge degree of freedom is found out to be nothing

but the reparametrisation invariance. Also, we matched the equation of motions at the first order and higher derivative level to show that even in the first order approach we are dealing, the same dynamics of the system persists.

Chapter 6

The Regge-Teitelboim model of Cosmology

After discussing the models from various fields of physics, the final model to test the first order formalism of this thesis will be from gravity. Specifically, we will consider the Regge-Teitelboim (RT) model. The RT model can be thought as an implementation of a gravity theory from the perspective of string theory. Gravity, in this case, is considered to be induced on a hypersurface which is evolving in a higher dimensional Minkowski space. According to the embedding theorem, the dimension of the embedding space should be at least $\frac{N(N+1)}{2}$, where N is the dimension of the hypersurface. So, the Einstein Hilbert action on a 4D hypersurface should be embedded in 10D Minkowski background [73]. The whole gravity theory, thereafter, is expressible with respect to the the local coordinates on the hypersurface.

These local coordinates, induced on the hypersurface, are considered to form a Minisuperspace in the higher dimensional embedding space. Unlike usual gravity theories where the metric is considered to be fundamental variable, these induced coordinates in this case are considered to be dynamical variables. Considering the FriedmannLematreRobertsonWalker (FRLW) metric induced on the hypersurface, the Einstein Hilbert action is computed. Apparently, the model is higher derivative but there appears a surface term which can be neglected as it has no effect on the equations of motion and the model becomes first order. But, in gravity theories these surface terms cannot be neglected so naively as they carry the information for the entropy of the system. So, keeping the surface term intact we consider the theory higher derivative version of the model.

For canonical analysis and hence exploring the gauge symmetries of this minisuperspace model we follow the first order formalism. It is found that there appears two first class and two second class constraints. Out of the two

first class constraints, one is primary first class constraint. The method of Banerjee et. al. shows that there is one independent gauge parameter [32]. The independent gauge symmetry is identified as the reparametrisation invariance.

Now, the next step is to quantise this model. The formal quantisation is performed by promoting the Poisson/Dirac brackets between the canonical variable to the level of commutator. Proper identification of the phase space is crucial to begin the quantisation procedure and it is done by removing all the constraints from the theory. The first class constraints are converted into second class by introduction of some ad hoc relations among the phase-space variables, known as gauge conditions. The number of gauge conditions are always equal to the number of independent first class constraints. The canonical variables, in the final reduced phase space, are identified. These canonical variables are the operators which span in some appropriate Hilbert space. The analogous Schroedinger equation in cosmology is known as the Wheeler-Dewitt (WDW) equation. The WDW equation in the fully reduced phase space is constructed and the conserved energy is identified.

6.1 Introduction of the model

The RT model considers a d -dimensional brane Σ which evolves in a N dimensional bulk space time with fixed Minkowski metric $\eta_{\mu\nu}$. The world volume swept out by the brane is a $d + 1$ dimensional manifold m defined by the embedding $x^\mu = X^\mu(\xi^a)$ where x^μ are the local coordinates of the background spacetime and ξ^a are local coordinates for m . The theory is given by the action functional

$$S[X] = \int_m d^{d+1}\xi \sqrt{-g} \left(\frac{\beta}{2} \mathcal{R} - \Lambda \right), \quad (6.1)$$

where β has the dimension $[L]^{1-d}$ and g is the determinant of the induced metric g_{ab} . Λ denotes cosmological constant and \mathcal{R} is the Ricci scalar. As has been already stated above, we will be confined to the minisuperspace cosmological model following from the RT model.

The standard procedure in cosmology is to assume that on the large scale the universe is homogeneous and isotropic. These special symmetries enable the 4 dimensional world volume representing the evolving universe to be embedded in a 5-dimensional Minkowski space time

$$ds^2 = -dt^2 + da^2 + a^2 d\Omega_3^2, \quad (6.2)$$

where $d\Omega_3^2$ is the metric for unit 3 sphere. To ensure the FRW case we take the following parametric representation for the brane

$$x^\mu = X^\mu(\xi^a) = (t(\tau), a(\tau), \chi, \theta, \phi), \quad (6.3)$$

$a(\tau)$ is known as the scale factor.

After ADM decomposition with space like unit normals ($N = \sqrt{\dot{t}^2 - \dot{a}^2}$ is the lapse function)

$$n_\mu = \frac{1}{N}(-\dot{a}, \dot{t}, 0, 0, 0), \quad (6.4)$$

the induced metric on the world volume is given by,

$$ds^2 = -N^2 d\tau^2 + a^2 d\Omega_3^2. \quad (6.5)$$

Now, one can compute the Ricci scalar which is given by

$$\mathcal{R} = \frac{6\dot{t}}{a^2 N^4} (a\ddot{a}\dot{t} - a\dot{a}\ddot{t} + N^2\dot{t}). \quad (6.6)$$

With these functions we can easily construct the Lagrangian density as

$$\mathcal{L} = \sqrt{-g} \left(\frac{\beta}{2} \mathcal{R} - \Lambda \right). \quad (6.7)$$

The Lagrangian in terms of arbitrary parameter τ can be written as [72]¹

$$L(a, \dot{a}, \ddot{a}, \dot{t}, \ddot{t}) = \frac{a\dot{t}}{N^3} (a\ddot{a}\dot{t} - a\dot{a}\ddot{t} + N^2\dot{t}) - Na^3 H^2. \quad (6.8)$$

Varying the action with respect to the field $a(\tau)$ we get the corresponding Euler Lagrange equation as

$$\frac{d}{d\tau} \left(\frac{\dot{a}}{\dot{t}} \right) = - \frac{N^2 (\dot{t}^2 - 3N^2 a^2 H^2)}{a\dot{t} (3\dot{t}^2 - N^2 a^2 H^2)}. \quad (6.9)$$

Note that the Lagrangian (6.8) contains higher derivative terms of the field a . However we can write it as [72]

$$L = -\frac{a\dot{a}^2}{N} + aN (1 - a^2 H^2) + \frac{d}{d\tau} \left(\frac{a^2 \dot{a}}{N} \right). \quad (6.10)$$

If we neglect the boundary term the resulting Lagrangian becomes usual first order one. As is well known the equation of motion is still given by (6.9). However the Hamiltonian analysis is facilitated if we retain the higher derivative term. Thus our Hamiltonian analysis will proceed from (6.8).

¹here $H^2 = \frac{\Lambda}{3\beta}$, a constant quantity

6.2 Hamiltonian analysis

This section contains the main results of this present chapter. As stated above our aim is to develop a new Hamiltonian analysis following from the Lagrangian (6.8) which is a second order theory. A Hamiltonian analysis of the same model has been discussed in [72] from the Ostrogradsky approach. We on the other hand adopt the equivalent first order formalism which has been demonstrated to be useful, specifically in treating the gauge invariances from the Hamiltonian point of view [29,57,68]. The point of departure is to convert (6.8) to a first order theory by defining the first derivative of a and t as additional fields and including the following constraints into the Lagrangian with the help of undetermined multipliers. These multipliers are then treated as new fields and the phase space is constructed by the entire set of fields along with their conjugate momenta defined in the usual way as is done for first order theories. Automatically primary constraints arise. The constraint analysis is then presented in detail. In addition to first class constraints the model also has second class constraints. The second class constraints are then strongly implemented by substituting the Poisson brackets by the corresponding Dirac brackets. Effectively the theory becomes a first class system with the symplectic algebra given by these Dirac brackets of which a complete list has been given.

The results derived so far are then used in two ways. First an analysis of the gauge invariances of the model has been done and its connection with the reparametrization invariance of the action has been discussed. Secondly, the gauge redundancy of the model has been eliminated by choosing an appropriate gauge. The final Dirac brackets have been used to reduce the phase space and indicate a formal quantization of the model.

In the equivalent first order formalism, we define the new fields as,

$$\begin{aligned}\dot{a} &= A \\ \dot{t} &= T,\end{aligned}\tag{6.11}$$

which also introduce new constraints in the system given by

$$\begin{aligned}A - \dot{a} &\approx 0 \\ T - \dot{t} &\approx 0.\end{aligned}\tag{6.12}$$

Now the higher derivative Lagrangian (6.8) is transformed to the first order Lagrangian where the constraints (6.12) are enforced through the Lagrange multipliers λ_a , and λ_t as

$$\begin{aligned}L' &= \frac{aT}{(T^2 - A^2)^{\frac{3}{2}}} \left(aT\dot{A} - aA\dot{T} + (T^2 - A^2)T \right) - (T^2 - A^2)^{\frac{1}{2}} a^3 H^2 \\ &\quad + \lambda_a (A - \dot{a}) + \lambda_t (T - \dot{t}).\end{aligned}\tag{6.13}$$

The Euler Lagrange equation of motion, obtained from the first order Lagrangian (6.13), by varying w.r.t. a , A , t , T , λ_a and λ_t , are respectively given by

$$\frac{2a(\dot{A}T^2 - AT\dot{T})}{(T^2 - A^2)^{\frac{3}{2}}} + \frac{T^2}{(T^2 - A^2)^{\frac{1}{2}}} - 3a^2H^2(T^2 - A^2)^{\frac{1}{2}} + \dot{\lambda}_a = 0 \quad (6.14)$$

$$\begin{aligned} & \frac{3a^2A(\dot{A}T^2 - AT\dot{T})}{(T^2 - A^2)^{\frac{5}{2}}} - \frac{d}{d\tau} \left(\frac{a^2T^2}{(T^2 - A^2)^{\frac{3}{2}}} \right) - \frac{a^2T\dot{T}}{(T^2 - A^2)^{\frac{3}{2}}} + \frac{aAT^2}{(T^2 - A^2)^{\frac{3}{2}}} + \\ & \frac{a^3AH^2}{(T^2 - A^2)^{\frac{1}{2}}} + \lambda_a = 0 \end{aligned} \quad (6.15)$$

$$\dot{\lambda}_t = 0 \quad (6.16)$$

$$\begin{aligned} & \frac{3a^2T(\dot{A}T^2 - AT\dot{T})}{(T^2 - A^2)^{\frac{5}{2}}} + \frac{2a^2\dot{A}T}{(T^2 - A^2)^{\frac{3}{2}}} - \frac{d}{d\tau} \left(\frac{a^2AT}{(T^2 - A^2)^{\frac{3}{2}}} \right) - \frac{a^2A\dot{T}}{(T^2 - A^2)^{\frac{3}{2}}} \\ & + \frac{2aT}{(T^2 - A^2)^{\frac{1}{2}}} - \frac{aT^3}{(T^2 - A^2)^{\frac{1}{2}}} + \lambda_t = 0 \end{aligned} \quad (6.17)$$

$$A - \dot{a} = 0 \quad (6.18)$$

$$T - \dot{t} = 0. \quad (6.19)$$

Eliminating the multipliers λ_a , and λ_t from the above equations we get back equation (6.9)

In the Hamiltonian formulation adopted here, the Lagrange multipliers are considered formally as independent fields and the momenta corresponding to them are introduced in the usual way. Here we denote the phase space coordinates by $q_\mu = a, t, A, T, \lambda_a, \lambda_t$ and their corresponding momenta as $\Pi_{q_\mu} = \Pi_a, \Pi_t, \Pi_A, \Pi_T, \Pi_{\lambda_a}, \Pi_{\lambda_t}$ with $\mu = 0, 1, 2, 3, 4, 5$. We adopt the usual definition

$$\Pi_{q_\mu} = \frac{\partial L'}{\partial \dot{q}_\mu}, \quad (6.20)$$

since the Lagrangian (6.13) is in the first order form. This is the point of departure of our Hamiltonian formulation from the Ostrogradsky formulation of [72].

From the definition of the phase space variables, we get the following pri-

mary constraints

$$\begin{aligned}
\Phi_1 &= \Pi_t + \lambda_t \approx 0 \\
\Phi_2 &= \Pi_a + \lambda_a \approx 0 \\
\Phi_3 &= \Pi_T + \frac{a^2 T A}{(T^2 - A^2)^{\frac{3}{2}}} \approx 0 \\
\Phi_4 &= \Pi_A - \frac{a^2 T^2}{(T^2 - A^2)^{\frac{3}{2}}} \approx 0 \\
\Phi_5 &= \Pi_{\lambda_t} \approx 0 \\
\Phi_6 &= \Pi_{\lambda_a} \approx 0.
\end{aligned} \tag{6.21}$$

The nonzero Poisson brackets between the primary constraints are computed as

$$\begin{aligned}
\{\Phi_1, \Phi_5\} &= 1 \\
\{\Phi_2, \Phi_3\} &= -\frac{2aTA}{(T^2 - A^2)^{\frac{3}{2}}} \\
\{\Phi_2, \Phi_4\} &= \frac{2aT^2}{(T^2 - A^2)^{\frac{3}{2}}} \\
\{\Phi_2, \Phi_6\} &= 1.
\end{aligned} \tag{6.22}$$

Taking the constraint combination

$$\Phi'_3 = T\Phi_3 + A\Phi_4, \approx 0, \tag{6.23}$$

we find that Φ'_3 commutes with all the constraints. The nonzero poisson brackets between the newly defined primary set of constraints $\Phi_1, \Phi_2, \Phi'_3, \Phi_4, \Phi_5, \Phi_6$, become

$$\begin{aligned}
\{\Phi_1, \Phi_5\} &= 1 \\
\{\Phi_2, \Phi_4\} &= \frac{2aT^2}{(T^2 - A^2)^{\frac{3}{2}}} \\
\{\Phi_2, \Phi_6\} &= 1.
\end{aligned} \tag{6.24}$$

We can write down the canonical Hamiltonian as

$$\begin{aligned}
H_{can} &= \Pi_{q_\mu} \dot{q}_\mu - L' \\
&= -\frac{aT^2}{(T^2 - A^2)^{\frac{1}{2}}} + (T^2 - A^2)^{\frac{1}{2}} a^3 H^2 - \lambda_a A - \lambda_t T.
\end{aligned} \tag{6.25}$$

The total Hamiltonian is given by

$$H_T = H_{can} + \Lambda_1 \Phi_1 + \Lambda_2 \Phi_2 + \Lambda_3 \Phi_3' + \Lambda_4 \Phi_4 + \Lambda_5 \Phi_5 + \Lambda_6 \Phi_6. \quad (6.26)$$

Here $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5, \Lambda_6$ are undetermined Lagrange multipliers. Preserving the primary constraints Φ_1, Φ_5, Φ_6 in time ($\{\Phi_i, H_T\} \approx 0$) the following Lagrange multipliers get fixed

$$\begin{aligned} \Lambda_5 &= 0 \\ \Lambda_1 &= T \\ \Lambda_2 &= A. \end{aligned}$$

Whereas, conservation of Φ_2 gives the following condition between Λ_4 and Λ_6

$$\frac{T^2}{(T^2 - A^2)^{\frac{1}{2}}} - 3a^2 H^2 (T^2 - A^2)^{\frac{1}{2}} + \Lambda_6 + \Lambda_4 \frac{2aT^2}{(T^2 - A^2)^{\frac{3}{2}}} = 0. \quad (6.27)$$

Time preservation of the constraint Φ_3' gives rise to the following secondary constraint

$$\Psi_1 = \frac{aT^2}{(T^2 - A^2)^{\frac{1}{2}}} - a^3 H^2 (T^2 - A^2)^{\frac{1}{2}} + \lambda_t T + \lambda_a A \approx 0. \quad (6.28)$$

Likewise, Φ_4 yields the following secondary constraint

$$\Psi_2 = \frac{aAT^2}{(T^2 - A^2)^{\frac{3}{2}}} - \frac{a^3 H^2 A}{(T^2 - A^2)^{\frac{1}{2}}} - \lambda_a \approx 0. \quad (6.29)$$

Nonzero brackets for Ψ_1 and Ψ_2 with the other constraints are given below,

$$\begin{aligned}
\{\Phi_2, \Psi_1\} &= -\frac{T^2 - 3a^2H^2(T^2 - A^2)}{(T^2 - A^2)^{\frac{1}{2}}} \\
\{\Phi_4, \Psi_1\} &= -\frac{aAT^2}{(T^2 - A^2)^{\frac{3}{2}}} - \frac{a^3H^2A}{(T^2 - A^2)^{\frac{1}{2}}} - \lambda_a \\
\{\Phi_3, \Psi_1\} &= -\frac{aT(2A^2 - T^2)}{(T^2 - A^2)^{\frac{3}{2}}} + \frac{a^3H^2T}{(T^2 - A^2)^{\frac{1}{2}}} - \lambda_t \\
\{\Phi_5, \Psi_1\} &= -T \\
\{\Phi_6, \Psi_1\} &= -A \\
\{\Phi_2, \Psi_2\} &= -\frac{AT^2}{(T^2 - A^2)^{\frac{3}{2}}} + \frac{3a^2H^2A}{(T^2 - A^2)^{\frac{1}{2}}} \\
\{\Phi_4, \Psi_2\} &= -\frac{aT^2(T^2 + 2A^2)}{(T^2 - A^2)^{\frac{5}{2}}} + \frac{a^3H^2T^2}{(T^2 - A^2)^{\frac{3}{2}}} \\
\{\Phi_6, \Psi_2\} &= 1.
\end{aligned} \tag{6.30}$$

Time preservation of Ψ_1 trivially gives $0 = 0$. A similar analysis involving Ψ_2 yields, on exploiting (6.27),

$$\begin{aligned}
\Lambda_4 &= -\frac{(T^2 - 3a^2H^2(T^2 - A^2))(T^2 - A^2)}{a(3T^2 - a^2H^2(T^2 - A^2))} \\
\Lambda_6 &= -\frac{(T^2 - 3a^2H^2(T^2 - A^2))(T^2 - a^2H^2(T^2 - A^2)^{\frac{1}{2}})}{(T^2 - A^2)^{\frac{1}{2}}(3T^2 - a^2H^2(T^2 - A^2))}.
\end{aligned} \tag{6.31}$$

The iterative procedure is thus closed and no more secondary constraints or other relations are generated.

The above analysis reveals that of all the Lagrange multipliers Λ_i , only Λ_1 remains undetermined in (6.26) signifying one independent gauge degree of freedom. It is interesting to note that this consistency is not always obvious in the Ostrogradsky formulation, as we have already mentioned in connection with the massive relativistic particle model [27].

We have now altogether eight primary and secondary constraints. Computation of the Poisson bracket between these constraints shows that only Φ'_3 is the first class constraint, whereas other seven constraints are apparently second class. The odd number of apparently second class constraints

signals the existence of additional first class constraints. Indeed, the new constraint combination

$$\Psi'_1 = \Psi_1 - \Lambda_1\Phi_1 - \Lambda_2\Phi_2 - \Lambda_4\Phi_4 - \Lambda_5\Phi_5 - \Lambda_6\Phi_6, \quad (6.32)$$

leads to a secondary first class constraint. So now we have two first class constraints Φ'_3, Ψ'_1 and six second class constraints $\Phi_1, \Phi_2, \Phi_4, \Phi_5, \Phi_6$ and Ψ_2 . The total number of phase space variables is twelve. The number of independent phase space variables is therefore $12 - (2 \times 2 + 6)$ i.e. 2. Later on we will explicitly identify these two variables. There is no enhancement of degrees of freedom as is customary for the higher derivative systems. This is consistent with the fact that (6.8) is not a genuine higher derivative system. Also, of the two first class constraints of the system, Φ'_3 is the sole primary first class constraint. The number of primary first class constraint matches with the residual number of undetermined multiplier in the total Hamiltonian. This fact will be important in the construction of the gauge generator.

To study gauge symmetry of the system we need to get rid of the second class constraints. This is done by the introduction of the Dirac brackets which enable us to set these constraints strongly zero. For simplicity of the calculation we remove them pair by pair. The Dirac bracket between the basic fields after removing $\Phi_1, \Phi_2, \Phi_5, \Phi_6$ remains same as their corresponding Poisson brackets. Solving $\Phi_1, \Phi_2, \Phi_5, \Phi_6$ the new constraint structure becomes

$$\begin{aligned} F_1 &= \Phi'_3 = T\Phi_3 + A\Phi_4 \approx 0 \\ F_2 &= \Psi'_1 = \Psi_1 - \Lambda_4\Phi_4 \approx 0 \\ S_1 &= \Phi_4 \approx 0 \\ S_2 &= \Psi_2 = \frac{aAT^2}{(T^2 - A^2)^{\frac{3}{2}}} - \frac{a^3AH^2}{(T^2 - A^2)^{\frac{1}{2}}} + \Pi_a \approx 0. \end{aligned} \quad (6.33)$$

For simplicity we use new notations $\{F_1, F_2\}$ and $\{S_1, S_2\}$ where, the first pair denotes the set of first class constraint and second pair denotes the remaining set of second class constraints. Some details of this reduction are given below.

To calculate Dirac brackets of the theory we first find out the Poisson brackets between the second class constraints which are written as

$$\Delta_{ij} = \{S_i, S_j\} = -\frac{aT^2(3T^2 - a^2H^2(T^2 - A^2))}{(T^2 - A^2)^{\frac{5}{2}}}\epsilon_{ij}, \quad (6.34)$$

with $\epsilon_{12} = 1$ and $i, j = 1, 2$. Using (2.15), we calculate the Dirac brackets between the basic fields which are given below (only the nonzero brackets are listed)

$$\begin{aligned}
\{a, A\}_D &= -\frac{(T^2 - A^2)^{\frac{5}{2}}}{aT^2(3T^2 - a^2H^2(T^2 - A^2))} \\
\{a, \Pi_a\}_D &= \frac{T^2 + 2A^2 - a^2H^2(T^2 - A^2)}{(3T^2 - a^2H^2(T^2 - A^2))} \\
\{a, \Pi_A\}_D &= -\frac{3aA}{3T^2 - a^2H^2(T^2 - A^2)} \\
\{a, \Pi_T\}_D &= \frac{a(T^2 + 2A^2)}{T(3T^2 - a^2H^2(T^2 - A^2))} \\
\{t, \Pi_t\}_D &= 1 \\
\{A, \Pi_a\}_D &= -\frac{A(T^2 - A^2)(T^2 - 3a^2H^2(T^2 - A^2))}{aT^2(3T^2 - a^2H^2(T^2 - A^2))} \\
\{A, \Pi_A\}_D &= \frac{2(T^2 - A^2)}{3T^2 - a^2H^2(T^2 - A^2)} \\
\{A, \Pi_T\}_D &= \frac{A(T^2 + 2A^2 - a^2H^2(T^2 - A^2))}{T(3T^2 - a^2H^2(T^2 - A^2))} \\
\{T, \Pi_T\}_D &= 1 \\
\{\Pi_a, \Pi_A\}_D &= -\frac{a(2T^4 + A^2T^2 + a^2H^2(T^2 - A^2)(9A^2 - 2T^2))}{(T^2 - A^2)^{\frac{3}{2}}(3T^2 - a^2H^2(T^2 - A^2))} \\
\{\Pi_a, \Pi_T\}_D &= \frac{aA(T^4 + 2T^2A^2 + a^2H^2(T^2 - A^2)(T^2 + 6A^2))}{T(T^2 - A^2)^{\frac{3}{2}}(3T^2 - a^2H^2(T^2 - A^2))} \\
\{\Pi_A, \Pi_T\}_D &= -\frac{a^2T(T^2 + 2A^2 - a^2H^2(T^2 - A^2))}{(T^2 - A^2)^{\frac{3}{2}}(3T^2 - a^2H^2(T^2 - A^2))}. \tag{6.35}
\end{aligned}$$

The introduction of the above Dirac brackets allows the second class pair $\{S_1, S_2\}$ to be strongly implemented. Note that the secondary first class constraint then becomes equal to the canonical Hamiltonian:

$$F_2 = \Psi_1 = -H_c = -T\Pi_t - \frac{T^2}{A}\Pi_a \approx 0. \tag{6.36}$$

Vanishing of the canonical Hamiltonian is a consequence of the reparametrisation invariance of the theory.

6.2.1 Construction of the gauge generator

The gauge generator is defined in (2.16) can be written as,

$$G = \epsilon_1 F_1 + \epsilon_2 F_2. \quad (6.37)$$

Here ϵ_1 and ϵ_2 are the gauge parameters. From equations (2.19-2.20) we find that $C_{122} = -1 = -C_{212}$ and $V_{12} = 1$ are the only nonzero structure functions. Now using equation (2.18) the following relation between the gauge parameters is obtained

$$\epsilon_1 = -\Lambda_3 \epsilon_2 - \dot{\epsilon}_2. \quad (6.38)$$

So here ϵ_2 may be chosen as the independent gauge parameter.

At this stage we observe that there is one independent parameter in the gauge generator (6.37). The conditions (2.39) following from the higher derivative nature is yet to be implemented. As has been mentioned earlier this may or may not impose additional restriction on the gauge parameters. The gauge transformations of the fields are given by

$$\delta a = \{a, G\}_D = -\epsilon_2 A \quad (6.39)$$

$$\delta t = -\epsilon_2 T \quad (6.40)$$

$$\delta A = \epsilon_1 A - \epsilon_2 \frac{(T^2 - 3a^2 H^2 (T^2 - A^2)) (T^2 - A^2)}{a (3T^2 - a^2 H^2 (T^2 - A^2))} \quad (6.41)$$

$$\delta T = \epsilon_1 T \quad (6.42)$$

After some calculation we find that

$$\frac{d}{d\tau} \delta a = \delta A \quad (6.43)$$

$$\frac{d}{d\tau} \delta t = \delta T. \quad (6.44)$$

So the constraints (2.39) hold identically for the present model and impose no new condition on the gauge parameters. We find therefore that there is only one independent gauge transformation which essentially is in conformation with the fact that there is only one independent primary first class constraint.

The gauge variations obtained from the Hamiltonian analysis can be exactly mapped to the reparametrization invariance of the model. Consider arbitrary infinitesimal change in the parameter $\tau \rightarrow \tau' = \tau + \sigma$. The action is invariant under this reparametrization. Now the fields transform as

$$\begin{aligned} \delta a &= -\sigma a \\ \delta t &= -\sigma t. \end{aligned} \quad (6.45)$$

These are identical with the gauge variations (6.39) and (6.40) of a and t if σ is identified with ϵ_2 . The equivalence of gauge invariances with the reparametrization invariance of the model is thus established.

6.2.2 Gauge fixing and formal quantization

After the reduction of phase space by the Dirac bracket procedure we are left with only the two first class constraints F_1 and F_2 . These first class constraints reflect the redundancy of the theory which are connected by gauge transformations. In the above analysis our focus was on the abstraction of the gauge degrees of freedom. We now elucidate a formal quantisation prescription. A gauge fixing is done and the appropriate WDW equation is written.

The choice of gauge is arbitrary subject to the conditions that they must reduce the first class constraints to second class. Also the constraint algebra should be nonsingular. As there are two first class constraints we need two gauge conditions. We take one of these to be the cosmic gauge

$$\varphi_1 = \sqrt{T^2 - A^2} - 1 \approx 0. \quad (6.46)$$

The name derives from the fact that the resultant metric becomes the usual FLRW metric. As the second gauge condition we take

$$\varphi_2 = T - \alpha a \approx 0. \quad (6.47)$$

where the constant α is chosen so that $\alpha \neq H$. The following calculations will show that these are appropriate gauge conditions.

As usual the gauge conditions are treated as additional constraints which make the first class constraints of the theory second class. For convenience, renaming the two first class constraints we write the complete set of constraints as

$$\Omega_1 = F_1 \quad (6.48)$$

$$\Omega_2 = F_2 \quad (6.49)$$

$$\Omega_3 = \varphi_1 \quad (6.50)$$

$$\Omega_4 = \varphi_2. \quad (6.51)$$

Modifying the algebra by the Dirac brackets corresponding to this second class system we will be able to put all the second class constraints ($\Omega_i, i = 1, 2, 3, 4$) to be strongly equal to zero. These will correspond to operator relations in the corresponding quantum theory.

Table 6.1: Constraint brackets

	Ω_1	Ω_2	Ω_3	Ω_4
Ω_1	0	0	-1	$-T$
Ω_2	0	0	$\frac{A(\alpha^2-3H^2)}{a(3\alpha^2-H^2)}$	$-\alpha A$
Ω_3	1	$-\frac{A(\alpha^2-3H^2)}{a(3\alpha^2-H^2)}$	0	$\frac{A}{\alpha a^5(3\alpha^2-H^2)}$
Ω_4	T	αA	$-\frac{A}{\alpha a^5(3\alpha^2-H^2)}$	0

Using the algebra (6.35) we can straightforwardly compute the algebra of the constraints Ω_i . The results are given in the following table
From the above table we can read off the matrix

$$\Delta_{ij} = \{\Omega_i, \Omega_j\} \quad (6.52)$$

Using the definition (2.15) we can calculate the final Dirac brackets. Nonzero Dirac brackets between the phase space variables are

$$\begin{aligned}
\{t, a\}^* &= \frac{1}{4\alpha a^3(\alpha^2 - H^2)} \\
\{t, A\}^* &= \frac{\alpha}{4a^2 A(\alpha^2 - H^2)} \\
\{t, T\}^* &= \frac{1}{4a^3(\alpha^2 - H^2)} \\
\{t, \Pi_a\}^* &= \frac{-4a^2\alpha^2 + 3}{4\alpha a A} \\
\{t, \Pi_A\}^* &= \frac{\alpha}{(\alpha^2 - H^2)} \\
\{t, \Pi_t\}^* &= 1 \\
\{t, \Pi_T\}^* &= \frac{-4a^2\alpha^2 + 3}{4a A(\alpha^2 - H^2)}. \quad (6.53)
\end{aligned}$$

With the introduction of the final Dirac brackets all the constraints (including the gauge conditions) become second class and strongly zero. We thus

have the following conditions on the phase space variables

$$\begin{aligned}
\Pi_T + \frac{a^2 T A}{(T^2 - A^2)^{\frac{3}{2}}} &= 0 \\
\Pi_A - \frac{a^2 T^2}{(T^2 - A^2)^{\frac{3}{2}}} &= 0 \\
-\Pi_t - \frac{T}{A} \Pi_a &= 0 \\
\frac{a A T^2}{(T^2 - A^2)^{\frac{3}{2}}} - \frac{a^2 A H^2}{(T^2 - A^2)^{\frac{1}{2}}} + \Pi_a &= 0 \\
\sqrt{T^2 - A^2} &= 1 \\
T - \alpha a &= 0
\end{aligned} \tag{6.54}$$

where use has been made of equations (6.33, 6.46, 6.47). From the final Dirac brackets (6.53) it is clear that only the pair (t, Π_t) is canonical. We thus identify this pair as the two independent phase space degrees of freedom found earlier by a standard count using the constraints of the system (see below 6.32). To develop a quantum theory it is necessary to write down the whole theory with respect to the canonical variables in the reduced phase. All the variables can be expressed in favour of (t, Π_t) by appropriately solving the constraints which are now strongly implemented. The result is,

$$\begin{aligned}
T &= \alpha a \\
A &= \sqrt{\alpha^2 a^2 - 1} \\
\Pi_A &= \alpha^2 a^4 \\
\Pi_T &= -\alpha a^3 \sqrt{\alpha^2 a^2 - 1} \\
\Pi_a &= -a^3 (\alpha^2 - H^2) \sqrt{\alpha^2 a^2 - 1}.
\end{aligned} \tag{6.55}$$

where a is expressed as

$$a = \left(\frac{\Pi_t}{\alpha(\alpha^2 - H^2)} \right)^{\frac{1}{4}}. \tag{6.56}$$

Thus we find that all the phase space variables except t are given as function of Π_t .

The passage from the classical to quantum theory proceeds in the usual way. The phase space variables are lifted to operators in some Hilbert space and the conditions (6.55, 6.56) are now treated as operator relations. The Dirac

brackets are promoted to commutators according to the prescription.

$$\{B, C\}^* \rightarrow \frac{1}{i\hbar}[B, C]. \quad (6.57)$$

The fundamental canonical algebra is thus (with $\hbar = 1$)

$$[t, t] = [\Pi_t, \Pi_t] = 0, \quad [t, \Pi_t] = i. \quad (6.58)$$

We next proceed to formulate the WDW equation for the universe governed by the Lagrangian (6.10). Before that we write down the first class constraint F_2 which is the canonical Hamiltonian as

$$F_2 = -H_{can} = -\frac{-A^2\Pi_t^2 + T^2\Pi_a^2}{A\Pi_a} = 0 \quad (6.59)$$

Considering the state vector $|\Psi\rangle$ in the appropriate Hilbert space, the WDW equation may be written as,

$$H_{can}|\Psi\rangle = 0. \quad (6.60)$$

Using the Schrodinger representation compatible with (6.58), we obtain,

$$\Pi_t = -i\frac{\partial}{\partial t} \quad (6.61)$$

Exploiting (6.59-6.61) and the expression for A given in (6.55) we obtain, after some algebra, the following WDW equation,

$$-\frac{\partial^2}{\partial t^2}|\Psi\rangle = \alpha^2 a^8 (\alpha^2 - H^2)^2 |\Psi\rangle. \quad (6.62)$$

Making a change of variables $\xi = \frac{\alpha^2}{H^2}$, the WDW equation may be reexpressed as,

$$-\frac{\partial^2}{\partial t^2}|\Psi\rangle = \xi(\xi - 1)^2 H^6 a^8 |\Psi\rangle. \quad (6.63)$$

The above equation exactly reproduces one piece of the bifurcated WDW equation found in the first item of [69]².

Furthermore, introducing the conserved ‘energy’ ω by,

$$\xi(\xi - 1)^2 H^6 a^8 = \omega^2 \quad (6.64)$$

²Note that the other part of the bifurcated WDW involving the ‘ a ’ variable is nonexistent in the present analysis. This is because here we have only one (configuration space) independent degree of freedom (i.e. t) instead of two variables (t and a) as occurs in [69]. This mismatch happens because, contrary to [69], the present analysis is done in a fully reduced space where all constraints are eliminated

we may reexpress (6.63) by the standard equation,

$$-\frac{\partial^2}{\partial t^2}|\Psi\rangle = \omega^2|\Psi\rangle. \quad (6.65)$$

The expression for the conserved energy ω in (6.64) matches with the form given in [69]. It is now possible to proceed with the quantisation as elaborated in [69].

Before concluding this section it is worthwhile to mention the efficacy of the gauge choice (6.47). While the first gauge condition (6.46) is the standard cosmic gauge, the second one (6.47) has not been considered earlier. We have shown that this simple choice (6.47) is a valid choice that yields the fully reduced space of the model. Also, at the quantum level, the WDW equation subjected to this gauge fixing reproduces the expression obtained earlier in [69].

6.3 Discussions

In this chapter, we have considered the Regge Teitelboim (RT) model in the minisuperspace version. The Einstein Hilbert action in 4D is expressed with respect to the minisuperspace variables. Apparently, the theory is a higher derivative theory, however, a surface term can be identified which actually contains the higher derivative term. As surface term plays important role in gravity so they cannot be neglected as in usual theories. Keeping this surface term, we considered the higher derivative version of the RT model and the first order formalism is followed to explore its gauge symmetries and quantise the model formally in the reduced phase space. There appears only one independent primary first class constraint and the corresponding gauge degrees of freedom is identified to be reparametrisation symmetry. To quantise the theory we remove the two first class constraints by considering two gauge conditions. One of the suitable gauge conditions we considered is the well known cosmic gauge while the other one is newly proposed. Under these conditions the two first class constraints become second class and the theory is quantised in the fully reduced phase space after removing the second class constraints. The Wheeler DeWitt equation which is the analogue of Schrodinger equation in cosmology is constructed. Its compatibility with the existing literature [69] is shown.

Chapter 7

Conclusions

Higher derivative theories, due to their importance in various fields of the physics, have become very important and interesting field to be investigated. The higher derivative theories contain many subtle aspects like the enhancement in the degrees of freedom in comparison with the first order theory. In no way, importance of the symmetry studies of these theories can be minimised. There are certain aspects of symmetry which show startling differences from first order theories. For example, it has been reported in the case of massive relativistic particle model with curvature that the number of independent primary first class constraints (PFC) apparently do not match with the number of independent gauge degrees of freedom [27]. This matching is known to exist in conventional theories. Notably, later study follows the Ostrogradski approach of Hamiltonian analysis of the higher derivative theories.

In this thesis, we have followed a first order formalism [29] rather than the Ostrogradski approach. We have developed a systematic algorithm of constructing the independent gauge generator using the technique developed by Banerjee et. al. for the usual first order theory. Also, we have shown the quantisation of a specific higher derivative theory.

We followed the first order approach for the Hamiltonian analysis of the higher derivative theories. The reason to consider the first order approach is that one can easily perform the canonical analysis of these higher derivative theories like the usual first order theories. Moreover, as far we have seen the proper identification of the gauge degrees of freedom can be done without any nontriviality. We already have seen that there appears a mismatch in the number of independent gauge degrees of freedom and number of independent primary first class constraints which indeed was solved by considering this first order approach [29]. According to this formalism we consider all the higher time derivatives of the dynamical variables as inde-

pendent fields and introduce them separately. Due to redefinition of the fields some interrelation between the fields appear which are treated as constraints. These constraints are added to the Lagrangian via some Lagrange multipliers. The Lagrangian is now a first order one. The gauge symmetries can now be abstracted via the Hamiltonian formulation. The gauge generator is constructed by using the first class constraints [33]. To get the independent gauge transformations we employ the equations derived in [32]. Other conditions appear strictly due to the higher derivative structure. Specifically, it has been noticed that this can give rise to some new relations which can be treated as constraints. After removing all the constraints by the Dirac bracket technique, now the theory can be quantised. The algorithm developed in this thesis has been applied to numerous pertinent models from different fields of current interest like the relativistic particle model with curvature, the extended Maxwell-Chern-Simon's model, the quantum charged rigid membrane model and the Regge-Teitelboim model. Below we give a summary of the results for these various models.

- The action of the relativistic particle model with curvature consists of a curvature term along with the usual relativistic term [6, 29, 30]. Existence of this curvature term makes the theory higher derivative. For the massive case, we have seen that there are two independent primary first class constraints. Whereas, there is only one independent gauge degree of freedom. This mismatch, which appears in the Ostrogradski formalism [27], was solved by considering the first order formalism. In the first order formalism, for the massive case, we got one independent primary first class constraint and one independent gauge symmetry. This gauge symmetry is identified as the diffeomorphism symmetry. Also, when we considered the mass shell condition $p_1^2 - m^2 = 0$ separately, the constraint structure changes. There appears one independent PFC. The Dirac brackets between the basic variables, in this case, also get modified.
- From the massive relativistic particle model with curvature, the massless version of this model is obtained by considering only the curvature term [29, 31]. Canonical analysis in the first order formalism is performed for this model. We have seen that, in this case, the constraint structure is different from the massive case. We have found five first class constraints, with no second class constraints, out of which two are primary. Using these first class constraints, we have constructed the Hamiltonian gauge generator. The method of Banerjee et. al. showed that in the gauge generator there are two independent gauge parameters. Restoring to a geometric approach we have identified these symmetries with the diffeomorphism and W -morphisms and explicitly demonstrated the W_3 -algebra. Note that previously the symmetries were demonstrated on-shell [45, 46] and a consistent

Hamiltonian analysis was lacking. By casting the equations of motion of this model in the Bossinesq form it was earlier demonstrated that the model is endowed with a larger symmetry group. It is interesting that our Hamiltonian method straightforwardly exhibits the full symmetry group and the W_3 -algebra is established in the Hamiltonian approach without taking recourse of the equations of motion.

- The BRST formulation for the higher derivative theories were lacking in the literature. For this reason, the BRST formulation for the massive and massless relativistic particle models were considered [54]. As the massive relativistic particle consists of the diffeomorphism symmetries, its BRST analogue is constructed. The massless model, on the other hand, has two gauge symmetries viz. diffeomorphism and W_3 symmetry. The nilpotent BRST and anti-BRST symmetries for these models have also been investigated. For the massless version, a remarkable feature for such symmetries comes out which is the manifestation of W_3 algebra. The BRST symmetries for all variables (except for the antighost variable) corresponding to diffeomorphism and W-morphism satisfy the W_3 -algebra. Likewise, apart from the ghost variable, the anti-BRST symmetry transformations for all other variables also satisfy the same W_3 algebra. Thus the full W_3 algebra for all variables is obtained by taking into account both BRST and anti-BRST transformations. The finite coordinate-dependent BRST (FCBRST) symmetry, which is a quantum mechanical analogue of finite field-dependent BRST (FFBRST), has been analyzed in full generality for higher derivative particle models. It has been shown that, although such a transformation is a symmetry of the effective action, it breaks the invariance of the generating functional of the path integral. The Jacobian of path integral measure changes nontrivially for FCBRST symmetry transformation. We have shown that FCBRST transformation with a suitable coordinate-dependent parameter changes the effective action from one gauge to another within a functional integral. Thus, FCBRST formulation is very useful to connect two different Greens functions for models of relativistic particles.
- The field theoretic model we considered is the Extended Maxwell-Chern-Simons theory. The Chern-Simons's piece considered here is a modified one and actually contains the higher derivative [55]. The field A_μ and their time derivatives are considered to be dynamical variables to apply the first order formalism. We see that there is only one independent PFC and the gauge generator also contains one independent gauge parameter. The Lagrangian gauge symmetry which is the $U(1)$ symmetry of the model is shown to match the gauge symmetry obtained by the Hamiltonian formulation in the first order approach.

- The next model we have considered is the quantum charged rigid membrane which is the modification of the Dirac's original membrane model of the electron [40]. At the Lagrangian level reparametrisation is the inherent gauge symmetry of this model. The action when expressed with respect to the coordinate variables on the membrane shows higher derivative nature. So, we have adopted the the first order formalism to explore its gauge symmetries. There appears one independent primary first class constraint and the corresponding gauge symmetry in the model is mapped to the reparametrisation invariance. Farther, to show the viability of this first order approach the equations of motion are shown to match for the higher derivative Lagrangian and the converted first order model.
- Finally, we considered a gravity model to show the applicability of the algorithm developed in this thesis for analysing the higher derivative theories. The model we have considered here is the Regge Teitelboim model of Cosmology. In the Regge Teitelboim model, a 4D gravity theory is considered to be induced on a hypersurface which is embedded in a 10D Minkowski spacetime. Unlike usual gravity theories where the metric is considered as independent variable, in this case, the embedding functions which are induced on the hypersurface are considered to be dynamical variables. The Lagrangian, however, is not a genuine higher derivative one as it contains a surface term which effectively makes the Lagrangian first order. We kept the surface term as in gravity theories they cannot be neglected. Hamiltonian formulation in the first order formulation is completed. The model has one independent gauge symmetry which is identified as reparametrisation invariance. As there appeared two independent first class constraints, to remove all the redundant degrees freedom, we considered two gauge conditions. One of the gauge condition is the cosmic gauge condition and the another one is newly proposed here [73]. The Wheeler DeWitt equation for this cosmological model is constructed and compatibility with the existing literature is shown.

In this thesis we have developed a systematic algorithm for the analysis of gauge symmetries of a higher derivative theory in the canonical framework. There are different approaches towards the Hamiltonian method of treating the higher derivative theories. We follow the equivalent first order approach where the momenta are defined in the usual way and the signature of the higher derivative nature is contained in the existence of the Lagrangian constraints.

The algorithm developed in the thesis has been applied to higher derivative models from various fields namely, we have considered the mechanical model, field theoretic model, gravity model etc. We have also connected our approach with the BRST formalism. In all these examples we have rederived

already well established results and resolved many issues which were not adequately discussed in the literature. Besides this, we have shown certain remarkable differences in the abstraction of gauge symmetries compared to usual first order systems. The number of independent gauge transformations is no longer strictly equal to the number of independent primary first class constraints of the higher derivative theory. The robustness and generality of our approach indicates its utility in the discussion and analysis of higher derivative singular system.

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